Inflation Expectations and Risk Premiums in an Arbitrage-Free Model of Nominal and Real Bond Yields†

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Abstract

Differences between yields on comparable-maturity U.S. Treasury nominal and real debt, the so-called breakeven inflation (BEI) rates, are widely used indicators of inflation expectations. However, better measures of inflation expectations could be obtained by subtracting inflation risk premiums from the BEI rates. We provide such decompositions using an estimated affine arbitrage-free model of the term structure that captures the pricing of both nominal and real Treasury securities. Our empirical results suggest that long-term inflation expectations have been well anchored over the past few years, and inflation risk premiums, although volatile, have been close to zero on average.

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From the perspective of monetary policy, just as important as the behavior of actual inflation is what households and businesses expect to happen to inflation in the future, particularly over the longer term. If people expect an increase in inflation to be temporary and do not build it into their longer-term plans for setting wages and prices, then the inflation created by a shock to oil prices will tend to fade relatively quickly. Some indicators of longer-term inflation expectations have risen in recent months, which is a significant concern for the Federal Reserve.

Federal Reserve Chairman Ben S. Bernanke (2008)

1 Introduction

In fulfilling their mandate for price stability, central banks around the world are keenly interested in the expectations of future inflation held by households and businesses. Such expectations are widely viewed as key determinants of future inflation, and long-run inflation expectations are considered a useful gauge of central bank credibility. There are two main sources for data on inflation expectations: surveys and financial markets. Policymakers consider the former source to be of somewhat limited use because surveys are typically conducted at a quarterly or, at best, a monthly frequency, so their information on expectations can be stale by the time of a policy meeting. Their infrequency also precludes using surveys to measure the immediate response of inflation expectations to discrete events, such as monetary policy actions or statements. In addition, surveys typically focus on the expectations of inflation over the next year or so, but given policy lags, central banks are interested in longer-term information as well, especially to help assess the credibility of their long-run inflation objectives.

In contrast to surveys, prices in financial markets can provide daily—even intraday—readings on inflation expectations at a wide range of horizons. The markets for nominal bonds, which have a fixed notional principal, and real bonds, which are directly indexed to overall price inflation, are the ones most closely followed for this purpose.[1] For example, the principal and coupon payments of U.S. Treasury inflation-protected securities (TIPS) vary with changes in the consumer price index (CPI). Differences between comparable-maturity nominal and real yields are known as breakeven inflation (BEI) rates. Like other central banks, the Federal Reserve closely monitors such BEI rates as high-frequency indicators of inflation expectations. However, as is widely appreciated, BEI rates are imperfect measures of

[1]Other financial instruments, such as inflation swaps, are also being introduced that may be useful for gauging inflation expectations.

inflation expectations because they also include compensation for inflation risk. That is, a BEI rate could rise if future inflation uncertainty rose or if investors required greater compensation for that uncertainty even if expectations for the future level of inflation remained unchanged. Obtaining a timely decomposition of BEI rates into inflation expectations and inflation risk premiums is important to central bankers, because they may wish to respond to a change in each component quite differently.

The decomposition of a BEI rate into inflation expectations and an inflation risk premium depends on the correlations between inflation and the unobserved stochastic discount factors of investors. This decomposition requires a model, and in this paper, we use an affine arbitrage-free (AF) model, which is the most widely used finance representation of the term structure.[2] These models specify the risk-neutral evolution of the underlying yield-curve factors as well as the dynamics of risk premiums under the key theoretical restriction that there are no residual opportunities for riskless arbitrage across maturities and over time. Following Duffie and Kan (1996), affine AF models have been particularly popular because yields are convenient linear functions of underlying latent factors (i.e., state variables that are unobserved by the econometrician) with factor loadings that can be calculated from a system of ordinary differential equations.

Unfortunately, affine AF models can exhibit very poor empirical time-series performance, especially when forecasting future yields (Duffee, 2002). In addition, there are many technical difficulties involved with the estimation of these models, which tend to be overparameterized and have numerous likelihood maxima that have essentially identical fit to the data but very different implications for economic behavior (Kim and Orphanides, 2005, and Duffee, 2008). Researchers have employed a variety of techniques to facilitate estimation including the imposition of additional model structure.[3] Notably, Christensen, Diebold, and Rudebusch (2007), henceforth CDR, impose general level, slope, and curvature factor loadings that are derived from the popular Nelson and Siegel (1987) yield curve. In this paper, we show that the resulting affine arbitrage-free Nelson-Siegel (AFNS) model can be readily estimated for a joint representation of nominal and real yield curves. Our estimated joint AFNS model for nominal and real yields describes the dynamics of the nominal and real stochastic discount factors, and with this model, we can decompose BEI rates of any maturity into inflation expectations and inflation risk premiums.

A sizable research literature has analyzed the inflation risk premium including, for ex-

[2]Other studies conduct this decomposition using historical inflation data; see Grishchenko and Huang (2008).

[3]For example, many researchers (e.g., Dai and Singleton, 2002) simply restrict parameters with small t-statistics in the first round of estimation to zero.

ample, Ang, Bekaert, and Wei (2008), who use a regime-switching AF model estimated on data for nominal yields and inflation. However, only a few papers have used U.S. data on real yields to estimate an AF model and decompose the BEI rate. Chen, Liu, and Cheng (2005) estimate a two-factor AF representation of nominal and real yields. Also, Hördahl and Tristani (2008) decompose BEI rates using an AF macro-finance model with monthly data on nominal and real yields, inflation, and the output gap, and D'Amico, Kim, and Wei (2008), henceforth DKW, estimate an affine AF model of nominal and real yields augmented with data on inflation and survey data on forecasts of short-term nominal interest rates. We compare our results to this earlier work.

The paper is structured as follows. In Sections 2 and 3, we estimate *separate* affine AF models for yields on nominal and real Treasury bonds, respectively. The resulting three-factor nominal model and two-factor real model are of some interest on their own and provide useful performance benchmarks for our *joint* model of nominal and real yields in Section 4. The separate models also provide an important input into the construction of that joint model. Indeed, based on the correlations among the factors of the separate models, we are able to identify a redundant factor, so we find a joint four-factor AF model fits both the nominal and real yield curves quite well. Section 5 then analyzes that estimated model's decomposition of BEI rates into inflation expectations and inflation risk premiums. Section 6 concludes.

2 An estimated AF model for nominal yields

In this section, we estimate an affine AF model on weekly data for nominal zero-coupon U.S. Treasury bond yields. An important preliminary step is to characterize the number and general form of the latent state variables. Researchers have typically found that three factors, often referred to as level, slope, and curvature, are sufficient to account for the time variation in the cross section of nominal Treasury yields (e.g., Litterman and Scheinkman, 1991). This characterization is supported by a principal component analysis of our weekly data set, which consists of Friday observations from January 6, 1995, to March 28, 2008, for eight maturities: three months, six months, one year, two years, three years, five years, seven years, and ten years.[4] Indeed, as shown in Table 1, 99.9 percent of the total variation in this set of yields is accounted for by the first three principal components. Furthermore, the loadings across the

[4]Our sample of nominal yields is relatively short because we are persuaded that there have been significant regime shifts in term structure behavior during the postwar period, not least of which stem from changes in the monetary policy rule linking short-term nominal interest rates and inflation. In addition, our sample of real yields is even more limited, so earlier data on nominal yields would be of limited value. Our data are obtained from http://www.federalreserve.gov/pubs/feds/2006/index.html and are described in Gürkaynak, Sack, and Wright (2007).

Maturity (in months)	First P.C.	Second P.C.	Third P.C.
3	-0.4174	-0.4227	0.5248
6	-0.4294	-0.3530	0.1630
12	-0.4279	-0.1952	-0.2527
24	-0.3937	0.0549	-0.4660
36	-0.3546	0.2101	-0.3721
60	-0.2888	0.3745	-0.0378
84	-0.2402	0.4541	0.2284
120	-0.1908	0.5148	0.4744
Explain	0.9417	0.0550	0.0030

Table 1: **First Three Principal Components in Nominal Yields.**
The loadings of yields of various maturities on the first three principal components are shown. The final row shows the proportion of all bond yield variability accounted for by each principal component. The data consist of weekly nominal zero-coupon U.S. Treasury bond yields from January 6, 1995, to March 28, 2008.

eight maturities for the first component are quite uniform; thus, like a level factor, a shock to this component will change all yields by a similar amount. The second component has negative loadings for short maturities and positive loadings for long ones; thus, like a slope factor, a shock to this component will steepen or flatten the yield curve. Finally, the third component has U-shaped factor loadings as a function of maturity and is naturally interpreted as a curvature factor.

These results help motivate our nominal AFNS model, which assumes that there are three latent state variables relevant for pricing nominal Treasury yields. The AFNS factors are identified as level, slope, and curvature by imposing the factor loadings from the Nelson and Siegel (1987) yield curve, which is commonly used by financial market practitioners and central banks due to its excellent fit to various real-world yield curves across countries and time; see Bank for International Settlements (2005), Diebold and Li (2006), Gürkaynak, Sack, and Wright (2007), and CDR (2008). CDR show that an AFNS model can closely fit the term structure of interest rates over time and forecasts well out of sample. CDR also show that the AFNS model can be estimated in a straightforward and robust fashion, unlike the canonical maximally flexible affine AF model, which, as noted in the introduction, is plagued with estimation difficulties.[5]

[5]Duffee (2008) describes the difficulties that require "a fairly elaborate hands-on estimation procedure." As an alternative strategy, DKW augment an AF model of nominal and real yields with data on inflation and survey data on forecasts of short-term nominal interest rates. However, adding inflation data to the model raises the thorny issue of reproducing the appropriate real-time information set of investors in light of lagged data releases and ex post data revisions. An advantage of our strategy is that our model only uses data from financial markets. In addition, adding survey data into the estimation raises questions about the congruency

The state vector of the three nominal AFNS model factors—level, slope, and curvature—is denoted as $X_t^N = (L_t^N, S_t^N, C_t^N)$. As discussed in CDR, the instantaneous nominal risk-free rate is assumed to be the sum of the level and slope factors:

$$r_t^N = L_t^N + S_t^N.$$

Also, the dynamics of the state variables under the risk-neutral (or Q) probability measure are given by the following system of stochastic differential equations:

$$\begin{pmatrix} dL_t^N \\ dS_t^N \\ dC_t^N \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda^N & \lambda^N \\ 0 & 0 & -\lambda^N \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t^N \\ C_t^N \end{pmatrix} dt + \Sigma^N \begin{pmatrix} dW_t^{Q,L^N} \\ dW_t^{Q,S^N} \\ dW_t^{Q,C^N} \end{pmatrix}, \quad \lambda^N > 0,$$

where W^Q is a standard Brownian motion in $\mathbf{R}^3$ and Σ^N is the volatility matrix.[6] An important restriction in this dynamic system is that the factor mean-reversion matrix (under the Q-measure) is constrained to take on a very simple form with λ^N as the only free parameter. CDR show that this AFNS structure implies that nominal zero-coupon yields with maturity τ at time t, $y_t^N(\tau)$, take the form:

$$y_t^N(\tau) = L_t^N + \left(\frac{1 - e^{-\lambda^N \tau}}{\lambda^N \tau}\right) S_t^N + \left(\frac{1 - e^{-\lambda^N \tau}}{\lambda^N \tau} - e^{-\lambda^N \tau}\right) C_t^N + \frac{A^N(\tau)}{\tau}.$$

That is, the three factors are given exactly the same level, slope, and curvature factor loadings as in the Nelson-Siegel (1987) yield curve. A shock to L_t^N affects yields at all maturities uniformly; a shock to S_t^N affects yields at short maturities more than long ones; and a shock to C_t^N affects midrange maturities most. Again, it is this identification of the general *role* of each factor, even though the factors themselves remain unobserved and the precise factor loadings depend on the estimated λ^N, that ensures the estimation of the AFNS model is straightforward and robust—unlike the maximally flexible affine AF model.

The yield function also contains a yield-adjustment term, $\frac{A^N(\tau)}{\tau}$, that is time-invariant and only depends on the maturity of the bond. CDR provide an analytical formula for this term, which under our identification scheme is entirely determined by the volatility matrix Σ^N. CDR find that allowing for a maximally flexible parameterization of the volatility matrix diminishes out-of-sample forecast performance, so we restrict Σ^N to be diagonal.

between the information sets of survey respondents and financial market participants.

[6]For identification, we fix the mean vector under the Q-measure at zero, which CDR show is without loss of generality.

The final element required for empirical implementation of an affine AF model is a specification of the price of risk. For tractable implementation, we employ the popular essentially affine risk premium specification introduced in Duffee (2002), which implies that the price of risk, Γ_t, depends on the state variables:

$$\Gamma_t = \gamma^0 + \gamma^1 X_t^N,$$

where $\gamma^0 \in \mathbf{R}^3$ and $\gamma^1 \in \mathbf{R}^{3\times 3}$ are unrestricted. The relationship between real-world yield curve dynamics under the P-measure and risk-neutral dynamics under the Q-measure is given by the measure change

$$dW_t^Q = dW_t^P + \Gamma_t dt.$$

Therefore, we can write the P-dynamics of the state variables as

$$dX_t^N = K^{P,N}(\theta^{P,N} - X_t^N)dt + \Sigma^N dW_t^P,$$

where both $K^{P,N}$ and $\theta^{P,N}$ are allowed to vary freely.[7]

We estimate this model using the Kalman filter, as detailed in the appendix. The Kalman filter provides consistent and efficient parameter estimates and easily handles missing data, which will be useful for combining nominal and real yields in the joint estimation.[8] Table 2 presents the estimated parameters for this model. The level factor is very persistent (with a rate of own mean reversion of only 0.100), while the slope and curvature factors revert to mean more quickly. Only a few of the off-diagonal elements in $K^{P,N}$ are significant, which is consistent with earlier work. For example, CDR find better out-of-sample forecast performance from an estimated model with a diagonal rather than an unrestricted $K^{P,N}$. However, we are largely interested in the separate nominal and real models in order to calculate the correlations between the associated nominal and real factors. These factors are essentially insensitive to any restrictions that might be placed on the $K^{P,N}$ matrix, so we simply employ the unrestricted, flexible version in the separate nominal and real models. However, for the joint model, where the estimates of the factor dynamics will affect the resulting decomposition of the BEI rate, we provide a thorough analysis of alternative dynamic specifications.

Summary statistics for the fitted errors of yields at each maturity of the estimated nominal AFNS model are given in the second and third columns of Table 3. With the exception of the

[7]The structure under the Q-measure places no restrictions on the dynamic drift components under the empirical P-measure beyond the requirement of constant volatility.

[8]Note that yields at each maturity have their own i.i.d. measurement error within the estimation process.

$K^{P,N}$	$K^{P,N}_{\cdot,1}$	$K^{P,N}_{\cdot,2}$	$K^{P,N}_{\cdot,3}$	$\theta^{P,N}$		Σ^N
$K^{P,N}_{1,\cdot}$	0.100	-0.068	0.0194	0.0629	$\Sigma^N_{1,1}$	0.00471
	(0.220)	(0.096)	(0.076)	(0.0071)		(0.00014)
$K^{P,N}_{2,\cdot}$	1.434	0.795	-0.997	-0.0219	$\Sigma^N_{2,2}$	0.00771
	(0.515)	(0.187)	(0.163)	(0.0113)		(0.00021)
$K^{P,N}_{3,\cdot}$	-0.757	0.177	0.736	-0.0113	$\Sigma^N_{3,3}$	0.02844
	(1.270)	(0.494)	(0.485)	(0.0112)		(0.00056)

Table 2: **Parameter Estimates for Nominal AFNS Model.**
The estimated parameters of the $K^{P,N}$ matrix, $\theta^{P,N}$ vector, and diagonal Σ^N matrix are shown for the AFNS model of nominal Treasury bond yields. The estimated value of λ^N is 0.5242 with a standard deviation of 0.0053. The maximum log-likelihood value is 32,311.36. The numbers in parentheses are estimated parameter standard deviations.

Maturity in months	Nominal AFNS model		Real AFNS model		Joint AFNS model	
Nom. yields	Mean	RMSE	Mean	RMSE	Mean	RMSE
3	-0.16	10.31	–	–	-0.26	10.38
6	0.00	0.00	–	–	0.00	0.00
12	1.63	6.12	–	–	1.76	6.18
24	2.18	4.07	–	–	2.30	4.15
36	0.00	0.00	–	–	0.00	0.00
60	-2.69	3.47	–	–	-2.88	3.81
84	0.22	2.37	–	–	-0.08	2.85
120	10.44	12.05	–	–	10.04	11.53
TIPS yields	Mean	RMSE	Mean	RMSE	Mean	RMSE
60	–	–	0.89	3.65	-4.04	10.19
72	–	–	0.27	1.06	-2.23	6.53
84	–	–	0.00	0.00	-0.99	3.19
96	–	–	-0.05	0.22	0.00	0.00
108	–	–	0.00	0.00	0.77	2.94
120	–	–	0.09	0.40	1.36	5.54

Table 3: **Summary Statistics for AFNS Models.**
The means and root mean squared errors (RMSE) of the fitted errors of the nominal, real, and joint AFNS models are shown. All numbers are measured in basis points. The nominal yields cover the period from January 6, 1995, to March 28, 2008, while the real TIPS yields cover the period from January 3, 2003, to March 28, 2008.

three-month and ten-year yields, the errors are quite low and indicate a reasonable overall fit to the cross-section of yields.[9]

[9] The three-month maturity is difficult to fit partly because the short end of the Treasury yield curve is buffeted by short-term idiosyncratic forces (Duffee, 1996).

3 An empirical AF model for real yields

In this section, we estimate an affine AF model for real zero-coupon U.S. Treasury bond yields derived from TIPS yields.[10] In the empirical literature on term structure modeling, the focus has been on nominal rather than real bond yields in part because of the relative scarcity of real debt. The U.S. Treasury first issued TIPS in 1997, but for several years after that initial issuance, the liquidity of the secondary TIPS market was greatly impaired by the small amount of securities outstanding and uncertainty about the Treasury's commitment to the program. Indeed, as described by Roush (2008), secondary TIPS market trading was very low at least into 2002, and DKW estimate that such illiquidity boosted TIPS yields by 1 to 2 percentage points. To avoid spurious quotes from the illiquid nascent years of this market, we begin our sample of TIPS yields in 2003; therefore, our real yield data cover the period from January 3, 2003, to March 28, 2008, and are measured at the end of business each Friday. In addition, due to the limited maturity range in the TIPS market, we only consider maturities of five, six, seven, eight, nine, and ten years.

As a preliminary analysis, Table 4 reports the loadings by maturity that correspond to the first three principal components for our sample of real yields. The first two components account for essentially all of the variation in the data, and these components have loadings that are consistent with level and slope interpretations. Given the limited range of available maturities for real yields, it is not surprising that a curvature factor is not needed. Therefore, we estimate an AFNS model for real yields with a state vector of two factors, denoted as $X_t^R = (L_t^R, S_t^R)$.[11] The instantaneous risk-free real rate is defined as the sum of the level and slope factors:

$$r_t^R = L_t^R + S_t^R.$$

The dynamics of these two factors under the Q-measure are given by the stochastic differential equations:

$$\begin{pmatrix} dL_t^R \\ dS_t^R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda^R \end{pmatrix} \begin{pmatrix} L_t^R \\ S_t^R \end{pmatrix} dt + \Sigma^R \begin{pmatrix} dW_t^{Q,L^R} \\ dW_t^{Q,S^R} \end{pmatrix}, \quad \lambda^R > 0,$$

where W^Q is a standard Brownian motion in $\mathbf{R}^2$ and Σ^R is a diagonal volatility matrix.[12]

By imposing this structure on the general affine model, real zero-coupon yields with

[10] Our data are obtained from the Federal Reserve Board of Governors; see Gürkaynak, Sack, and Wright (2008) and the website http://www.federalreserve.gov/pubs/feds/2008/index.html.

[11] As an alternative, we also estimated a three-factor real AFNS model and found it to be overparameterized.

[12] Again, for identification, we fix the mean vector under the Q-measure at zero.

Maturity	First	Second	Third
60	-0.5361	0.6126	0.5113
72	-0.4675	0.2180	-0.3451
84	-0.4113	-0.0644	-0.4962
96	-0.3647	-0.2730	-0.2759
108	-0.3257	-0.4323	0.1090
120	-0.2928	-0.5583	0.5341
Explain	0.9766	0.0232	0.0002

Table 4: **First Three Principal Components in Real Yields.**
The loadings of yields of various maturities on the first three principal components are shown. The final row shows the proportion of all bond yield variability accounted for by each principal component. The data consist of weekly real zero-coupon bond yields from January 3, 2003, to March 28, 2008.

maturity τ at time t, $y_t^R(\tau)$, are given by

$$y_t^R(\tau) = L_t^R + \left(\frac{1 - e^{-\lambda^R \tau}}{\lambda^R \tau}\right) S_t^R + \frac{A^R(\tau)}{\tau},$$

which has Nelson-Siegel factor loadings for the level and slope factors and a maturity-dependent yield-adjustment term, $\frac{A^R(\tau)}{\tau}$, as described in CDR. As above, we only consider diagonal volatility matrices. As before, we employ the essentially affine risk premium specification:

$$\Gamma_t = \gamma^0 + \gamma^1 X_t^R,$$

where $\gamma^0 \in \mathbf{R}^2$ and $\gamma^1 \in \mathbf{R}^{2\times 2}$ are unrestricted. The same relationship between real-world and risk-neutral dynamics applies; therefore, we can write the P-dynamics of the state variables as

$$dX_t^R = K^{P,R}(\theta^{P,R} - X_t^R)dt + \Sigma^R dW_t^P,$$

where both $K^{P,R}$ and $\theta^{P,R}$ are allowed to vary freely.

Table 5 presents the estimated parameters for the dynamics of the two state variables based on the Kalman filter to obtain maximum likelihood estimates. Both factors revert to mean fairly quickly. The real level factor has an estimated volatility similar to that of the nominal level factor, but the real slope factor is estimated to be twice as volatile as the nominal slope factor. The fitted errors of this real AFNS model are reported in the fourth and fifth columns of Table 3. Their small size indicates that two factors are sufficient to model the variation in our TIPS yield sample, which is consistent with the principal component analysis.

$K^{P,R}$	$K^{P,R}_{\cdot,1}$	$K^{P,R}_{\cdot,2}$	$\theta^{P,R}$		Σ^R
$K^{P,R}_{1,\cdot}$	1.497	0.162	0.0294	$\Sigma^R_{1,1}$	0.00510
	(0.696)	(0.150)	(0.0020)		(0.00011)
$K^{P,R}_{2,\cdot}$	1.903	0.672	-0.0328	$\Sigma^R_{2,2}$	0.01497
	(2.720)	(0.574)	(0.0152)		(0.00071)

Table 5: **Parameter Estimates for Real AFNS Model.**
The estimated parameters of the $K^{P,R}$ matrix, $\theta^{P,R}$ vector, and diagonal Σ^R matrix are shown for the AFNS model of TIPS yields. The estimated value of λ^R is 0.3613 with a standard deviation of 0.0035. The maximum log-likelihood value is 12,359.09. The numbers in parentheses are estimated parameter standard deviations.

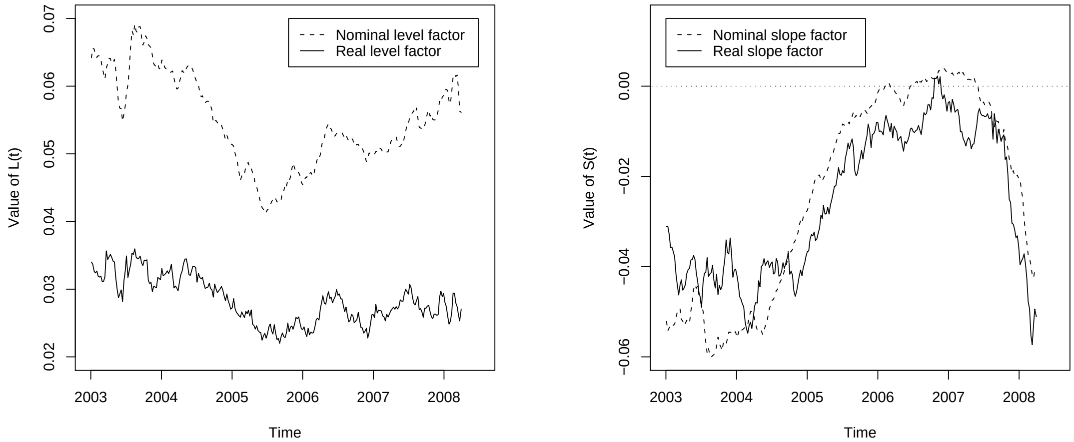
4 A joint AF model for nominal and real yields

An attractive feature of the AFNS model is that it can be extended to incorporate as many factors as required. For example, Christensen and Lopez (2008) estimate a joint AFNS model that accounts for the standard three Treasury yield factors and two additional factors accounting for corporate credit spread dynamics. In this section, we estimate a joint AFNS model that combines the separate nominal and real yield models presented above. Figure 1 compares the estimated paths of the two real yield curve factors from the real AFNS model to the paths of the corresponding nominal yield curve factors from the nominal AFNS model. The correlation between the two level factors is 0.90, while the slope factors have a correlation of 0.92. Given these high correlations, it is tempting to use just three factors to model the variation in both sets of bond yields; however, as described in the appendix, we found that a three-factor joint AFNS model was too restrictive to fit both nominal and real yields. Instead, we only impose the assumption of a common slope factor across the nominal and real yields. Therefore, our joint model has four factors: a real level factor (L^R_t) that is specific to TIPS yields only, a nominal level factor (L^N_t) for nominal yields, and common slope and curvature factors. (The curvature factor, of course, is only needed for fitting the nominal yields.)

The state vector of joint AFNS model factors is denoted as $X^J_t = (L^N_t, S_t, C_t, L^R_t)$, and the instantaneous nominal and real risk-free rates are defined by:

$$\begin{aligned} r^N_t &= L^N_t + S_t, \\ r^R_t &= L^R_t + \alpha^R S_t. \end{aligned}$$

The differential scaling of real rates to the common slope factor is captured by the parameter α^R. To preserve the Nelson-Siegel factor loading structure in the nominal yield function, the



(a) Estimated level factors. (b) Estimated slope factors.

Figure 1: **Estimated Nominal and Real Level and Slope Factors.**
The estimated level and slope factors from the two-factor real AFNS model are shown with the level and slope factors from the three-factor nominal AFNS model.

Q-dynamics of the state variables are given by the stochastic differential equations:

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} dt + \Sigma^J \begin{pmatrix} dW_t^{Q,L^N} \\ dW_t^{Q,S} \\ dW_t^{Q,C} \\ dW_t^{Q,L^R} \end{pmatrix}. \quad (1)$$

Based on these dynamics, nominal Treasury zero-coupon yields are

$$y_t^N(\tau) = L_t^N + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t + \frac{A^N(\tau)}{\tau},$$

and real zero-coupon yields are

$$y_t^R(\tau) = L_t^R + \alpha^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \alpha^R \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t + \frac{A^R(\tau)}{\tau}.$$

Again, details of the yield-adjustment terms are in CDR. Using the essentially affine risk premium specification, the implied measure change is given by

$$dW_t^Q = dW_t^P + \Gamma_t dt,$$

where $\Gamma_t = \gamma^{J,0} + \gamma^{J,1} X_t^J$, $\gamma^{J,0} \in \mathbf{R}^4$, and $\gamma^{J,1} \in \mathbf{R}^{4\times 4}$. The resulting four-factor AFNS model has P-dynamics given by

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,J} & \kappa_{12}^{P,J} & \kappa_{13}^{P,J} & \kappa_{14}^{P,J} \\ \kappa_{21}^{P,J} & \kappa_{22}^{P,J} & \kappa_{23}^{P,J} & \kappa_{24}^{P,J} \\ \kappa_{31}^{P,J} & \kappa_{32}^{P,J} & \kappa_{33}^{P,J} & \kappa_{34}^{P,J} \\ \kappa_{41}^{P,J} & \kappa_{42}^{P,J} & \kappa_{43}^{P,J} & \kappa_{44}^{P,J} \end{pmatrix} \left(\begin{pmatrix} \theta_1^{P,J} \\ \theta_2^{P,J} \\ \theta_3^{P,J} \\ \theta_4^{P,J} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right) dt + \Sigma^J \begin{pmatrix} dW_t^{P,L^N} \\ dW_t^{P,S} \\ dW_t^{P,C} \\ dW_t^{P,L^R} \end{pmatrix},$$

where Σ^J is diagonal.

As alluded to earlier, the specification of the P-dynamics is an important element in determining the model's decomposition of BEI rates into inflation expectations and risk premiums. Therefore, we conduct a careful evaluation of various model specifications, as summarized in Table 6. The first column of this table describes the 13 alternative specifications considered. Specification (1) at the top corresponds to an unrestricted 4×4 mean-reversion matrix $K^{P,J}$, which provides maximum flexibility in fitting the data. We then pare down this matrix using a general-to-specific strategy that restricts the least significant parameter (as measured by ratio of the parameter value to its standard error) to zero and then re-estimate the model. Therefore, specification (2) sets $\kappa_{31}^{P,J} = 0$, so it has one fewer estimated parameters. Specification (3) sets this parameter and $\kappa_{32}^{P,J}$ both equal to zero. This strategy of eliminating the least significant coefficients continues to the final specification (13), which has a diagonal $K^{P,J}$ matrix.

Each estimated specification is listed with its log likelihood ($\log L$), its number of estimated parameters (k), and the p-value from a likelihood ratio test of the hypothesis that it differs from the specification with one more free parameter—that is, comparing specification (s) with specification ($s-1$). We also report two information criteria commonly used for model selection: the Akaike information criterion, which is defined as AIC$= -2\log L + 2k$, and the Bayes information criterion, which is defined as BIC$= -2\log L + k \log T$, where T is the number of data observations (see e.g., Harvey, 1989).[13] These information criteria are minimized by specifications (8) and (9) (the boldface entries in the rightmost columns), which are thus our favored models. Notably, the unrestricted specification (1) appears overparameterized, and the diagonal specification (13) appears too parsimonious. The likelihood ratio test also suggests that (at the 10-percent level) specification (8) is a parsimonious model that still provides as good a fit to the data as the maximally flexible unrestricted specification. Therefore, we select specification (8) as our preferred joint AFNS model.

[13]We have 691 nominal yield and 273 real yield weekly observations. We interpret T as referring to the longest data series and fix it at 691.

Alternative specifications	Goodness of fit statistics				
	$\log L$	k	p-value	AIC	BIC
(1) Unrestricted $K^{P,J}$	42354.0	40	n.a	-84628.1	-84446.5
(2) $\kappa_{31}^{P,J} = 0$	42354.0	39	0.89	-84630.0	-84453.1
(3) $\kappa_{31}^{P,J} = \kappa_{32}^{P,J} = 0$	42353.9	38	0.69	-84631.9	-84459.4
(4) $\kappa_{31}^{P,J} = \kappa_{32}^{P,J} = \kappa_{13}^{P,J} = 0$	42353.9	37	0.69	-84633.7	-84465.8
(5) $\kappa_{31}^{P,J} = ... = \kappa_{34}^{P,J} = 0$	42353.6	36	0.46	-84635.2	-84471.8
(6) $\kappa_{31}^{P,J} = ... = \kappa_{12}^{P,J} = 0$	42353.2	35	0.39	-84636.4	-84477.6
(7) $\kappa_{31}^{P,J} = ... = \kappa_{24}^{P,J} = 0$	42352.7	34	0.31	-84637.4	-84483.1
(8) $\kappa_{31}^{P,J} = ... = \kappa_{43}^{P,J} = 0$	42352.5	33	0.50	**-84639.0**	-84489.2
(9) $\kappa_{31}^{P,J} = ... = \kappa_{41}^{P,J} = 0$	42350.7	32	0.06	-84637.5	**-84492.3**
(10) $\kappa_{31}^{P,J} = ... = \kappa_{42}^{P,J} = 0$	42345.9	31	< 0.01	-84629.8	-84489.1
(11) $\kappa_{31}^{P,J} = ... = \kappa_{21}^{P,J} = 0$	42335.4	30	< 0.01	-84610.9	-84474.7
(12) $\kappa_{31}^{P,J} = ... = \kappa_{14}^{P,J} = 0$	42326.5	29	< 0.01	-84595.0	-84463.4
(13) $\kappa_{31}^{P,J} = ... = \kappa_{23}^{P,J} = 0$	42303.3	28	< 0.01	-84550.7	-84423.6

Table 6: **Evaluation of Alternative Specifications of Joint AFNS Model.** Thirteen alternative estimated specifications of the joint AFNS model are evaluated. Each specification is listed with its log likelihood ($\log L$), number of parameters (k), the p-value from a likelihood ratio test of the hypothesis that the specification differs from the one directly above that has one more free parameter. The information criteria (AIC and BIC) are also reported, and their minimum values are given in boldface.

Table 7 contains the estimated parameters of this preferred specification (8). Note that the off-diagonal elements in the estimated $K^{P,J}$ matrix (excluding the seven zero restrictions) are highly statistically significant. The mean and volatility parameters for the three nominal factors and the estimated value of λ are very similar to those reported in Table 2 for the three-factor nominal AFNS model. Based on these results, we anticipate the fit of the nominal yields and the estimated paths of the three nominal yield risk factors to be very similar across these two models. Indeed, as shown in Table 3, which contains summary statistics for the fitted errors of the joint model, there is no discernible difference in fit between the joint model and the nominal model for the eight maturities of nominal yields. Table 3 does report a worse fit of the joint model relative to the two-factor model for real yields. However, the difference in fit appears to be reasonable in that the estimated real factors behave similarly. That is, the correlation between the estimated real level factors from the joint and real AFNS models is 0.86. The correlation between the estimated real slope factor from the real AFNS model and the joint AFNS model-implied real slope factor ($\alpha^R S_t$) is 0.92. The close connections between these factors provide further support for the joint model.

$K^{P,J}$	$K^{P,J}_{\cdot,1}$	$K^{P,J}_{\cdot,2}$	$K^{P,J}_{\cdot,3}$	$K^{P,J}_{\cdot,4}$	$\theta^{P,J}$		Σ^J
$K^{P,J}_{1,\cdot}$	1.305	0	0	-1.613	0.06317	$\Sigma^J_{1,1}$	0.00447
	(0.277)			(0.517)	(0.00112)		(0.00016)
$K^{P,J}_{2,\cdot}$	1.559	0.828	-1.044	0	-0.01991	$\Sigma^J_{2,2}$	0.00756
	(0.504)	(0.164)	(0.141)		(0.00862)		(0.000231)
$K^{P,J}_{3,\cdot}$	0	0	0.884	0	-0.00969	$\Sigma^J_{3,3}$	0.02926
			(0.382)		(0.00653)		(0.00058)
$K^{P,J}_{4,\cdot}$	-1.531	-0.364	0	1.645	0.03455	$\Sigma^J_{4,4}$	0.00413
	(0.643)	(0.110)		(0.724)	(0.00246)		(0.00014)

Table 7: **Parameter Estimates for Joint AFNS Model.**
The estimated parameters of the $K^{P,J}$ matrix, $\theta^{P,J}$ vector, and diagonal Σ^J matrix are shown for the AFNS model of nominal and real yields. The estimated value of λ is 0.5319 with a standard deviation of 0.0052, while α^R is estimated to be 0.6777 with a standard deviation of 0.0064. The numbers in parentheses are the estimated parameter standard deviations.

5 Inflation expectations and inflation risk premiums

In this section, we decompose the BEI rates into inflation expectations and inflation risk premiums. We start with a theoretical discussion of how an AF model of nominal and real yields can produce this decomposition, and then we present the empirical decomposition provided by our estimated joint AFNS model. Finally, we compare our results to others in the literature.

5.1 Theoretical discussion

To describe the connections among nominal and real yields and inflation within our modeling framework, it is convenient to work in continuous time (see Cochrane 2001 for a primer). We first define the nominal and real stochastic discount factors, denoted M_t^N and M_t^R, respectively. The no-arbitrage condition enforces a consistency of pricing for any security over time. Specifically, the price of a nominal bond that pays one dollar at time τ and the price of a real bond that pays one unit of the consumption basket at time τ must satisfy

$$P_t^N(\tau) = E_t^P\left[\frac{M_{t+\tau}^N}{M_t^N}\right] \quad \text{and} \quad P_t^R(\tau) = E_t^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right].$$

Given their payment structure, the no-arbitrage condition also requires a consistency between the prices of real and nominal bonds such that the price of the consumption basket, denoted as the overall price level Q_t, is the ratio of the nominal and real stochastic discount factors:

$$Q_t = \frac{M_t^R}{M_t^N}.$$

We assume that the nominal and real stochastic discount factors have the standard dynamics given by

$$\begin{aligned} dM_t^N/M_t^N &= -r_t^N dt - \Gamma_t' dW_t^P, \\ dM_t^R/M_t^R &= -r_t^R dt - \Gamma_t' dW_t^P. \end{aligned}$$

Then, by Ito's lemma, the dynamic evolution of Q_t is given by

$$\begin{aligned} dQ_t &= \frac{1}{M_t^N} dM_t^R - \frac{M_t^R}{(M_t^N)^2} dM_t^N + \frac{1}{2} \begin{pmatrix} dM_t^R & dM_t^N \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{(M_t^N)^2} \\ -\frac{1}{(M_t^N)^2} & 2\frac{M_t^R}{(M_t^N)^3} \end{pmatrix} \begin{pmatrix} dM_t^R \\ dM_t^N \end{pmatrix} \\ &= (r_t^N - r_t^R) Q_t dt. \end{aligned}$$

Thus, with the absence of arbitrage, the instantaneous growth rate of the price level is equal to the difference between the instantaneous nominal and real risk-free rates. (That is, there is no risk premium for the instantaneous rates, and the Fisher equation applies.) Furthermore, by Ito's lemma,

$$dln\,(Q_t) = \frac{1}{Q_t} dQ_t - \frac{1}{2}\frac{1}{Q_t^2} dQ_t^2 = (r_t^N - r_t^R) dt.$$

By integrating both sides and taking exponentials, we can express the price level at time t+τ as

$$Q_{t+\tau} = Q_t e^{\int_t^{t+\tau} (r_s^N - r_s^R) ds}.$$

The connection between nominal and real zero-coupon yields and expected inflation can be readily expressed. Namely, we decompose the price of the nominal zero-coupon bond as

$$\begin{aligned} P_t^N(\tau) &= E^P\left[\frac{M_{t+\tau}^N}{M_t^N}\right] \\ &= E^P\left[\frac{M_{t+\tau}^R/Q_{t+\tau}}{M_t^R/Q_t}\right] = E^P\left[\frac{M_{t+\tau}^R}{M_t^R}\frac{Q_t}{Q_{t+\tau}}\right] \\ &= E^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right] \times E^P\left[\frac{Q_t}{Q_{t+\tau}}\right] + cov\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{Q_t}{Q_{t+\tau}}\right] \\ &= E^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right] \times E^P\left[\frac{Q_t}{Q_{t+\tau}}\right] \times \left(1 + \frac{cov\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{Q_t}{Q_{t+\tau}}\right]}{E^P\left[\frac{M_{t+\tau}^R}{M_t^R}\right] \times E^P\left[\frac{Q_t}{Q_{t+\tau}}\right]}\right). \end{aligned}$$

Converting this price into a yield-to-maturity, we obtain

$$y_t^N(\tau) = y_t^R(\tau) + \pi_t^e(\tau) + \phi_t(\tau),$$

where the market-implied rate of inflation expected at time t from the period t to $t+\tau$ is

$$\pi_t^e(\tau) = -\frac{1}{\tau} \ln E^P \left[\frac{Q_t}{Q_{t+\tau}} \right] = -\frac{1}{\tau} \ln E_t^P \left[e^{-\int_t^{t+\tau} (r_s^N - r_s^R) ds} \right],$$

and the corresponding inflation risk premium is

$$\phi_t(\tau) = -\frac{1}{\tau} \ln \left(1 + \frac{cov\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{Q_t}{Q_{t+\tau}} \right]}{E^P\left[\frac{M_{t+\tau}^R}{M_t^R} \right] \times E^P\left[\frac{Q_t}{Q_{t+\tau}} \right]} \right).$$

This last equation highlights that the inflation risk premium can be positive or negative. It will be positive if and only if

$$cov\left[\frac{M_{t+\tau}^R}{M_t^R}, \frac{Q_t}{Q_{t+\tau}} \right] < 0.$$

That is, the riskiness of nominal bonds depends on the covariance between the real stochastic discount factor and inflation. We observe positive inflation risk premiums if the real discount factor tends to be high (i.e., in a structural model, marginal utility is high) at the same time that price inflation is high (i.e., purchasing power is low).

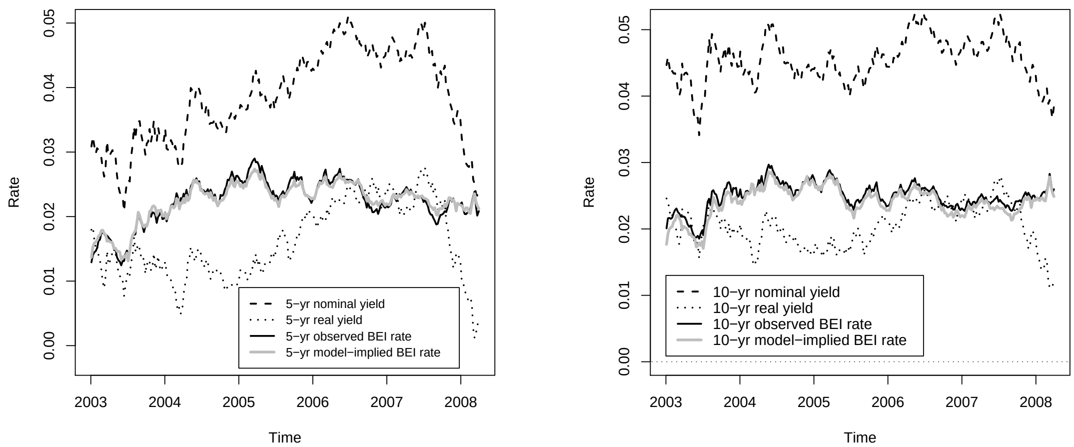
Finally, the BEI rate is defined as

$$BEI_t(\tau) \equiv y_t^N(\tau) - y_t^R(\tau) \quad = \quad \pi_t^e(\tau) + \phi_t(\tau).$$

Namely, the BEI rate is the difference between nominal and real yields and can be decomposed into the sum of expected inflation and the inflation risk premium.

5.2 Empirical results

We now turn to our estimated model. Figure 2 displays the five- and ten-year nominal and real zero-coupon yields and their differences at each maturity—i.e., the associated observed BEI rates. Both five- and ten-year BEI rates increased a bit during the first two years of our sample, but since 2004, they have changed little on balance. Figure 2 also compares these observed BEI rates to comparable-maturity model-implied BEI rates, which are calculated as the differences between the fitted nominal and real yields from the estimated joint AFNS model. The small differences between the observed and model-implied BEI rates reflect the



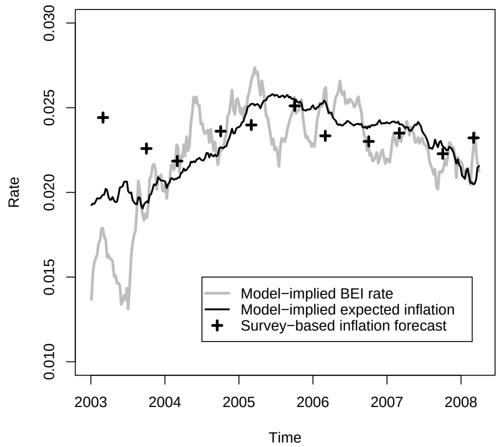
(a) Five-year maturity (b) Ten-year maturity

Figure 2: **Nominal and Real Yields and BEI Rates.**
Data on five- and ten-year nominal and real zero-coupon Treasury yields are plotted with the associated BEI rates and the implied BEI rates from the joint AFNS model.

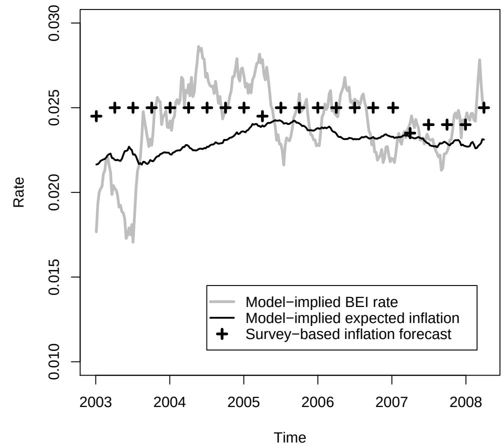
overall good fit of this model.

The joint AFNS model also allows us to decompose the BEI rate into inflation expectations and the inflation risk premium at various horizons. Given the estimated parameters in Table 7 and the estimated paths of the four state variables, the model-implied average five- and ten-year expected inflation series are illustrated in Figure 3. The five-year measure varied from 1.93 percent to 2.57 percent, and the ten-year measure from 2.16 percent to 2.42 percent. These ranges suggest that long-run inflation expectations were fairly well-anchored during our sample period.

The model's measures of inflation expectations are generated using only nominal and real yields without any data on inflation or inflation expectations. To provide some independent indication of accuracy, Figure 3 also plots survey-based measures of expectations of CPI inflation, which are obtained from the Blue Chip Consensus survey at the five-year horizon and from the Survey of Professional Forecasters at the ten-year horizon. The relatively close match between the model-implied and the survey-based measures of inflation expectations provides further support for the model's decomposition of the BEI rate. Note that the largest differences in Figure 3 occur for the five-year horizon during the first half of 2003 with the model-implied measure well below the survey-based one. This pattern is consistent with some remaining residual liquidity deficiencies in the TIPS market, which would hold down bond prices, boost real yields, and lead to an understatement of model-implied inflation



(a) Five-year horizon.



(b) Ten-year horizon.

Figure 3: **BEI Rates and Expected Inflation.**
The five- and ten-year BEI rates and average expected inflation rates that are implied from the joint AFNS model are plotted along with survey-based measures of inflation expectations.

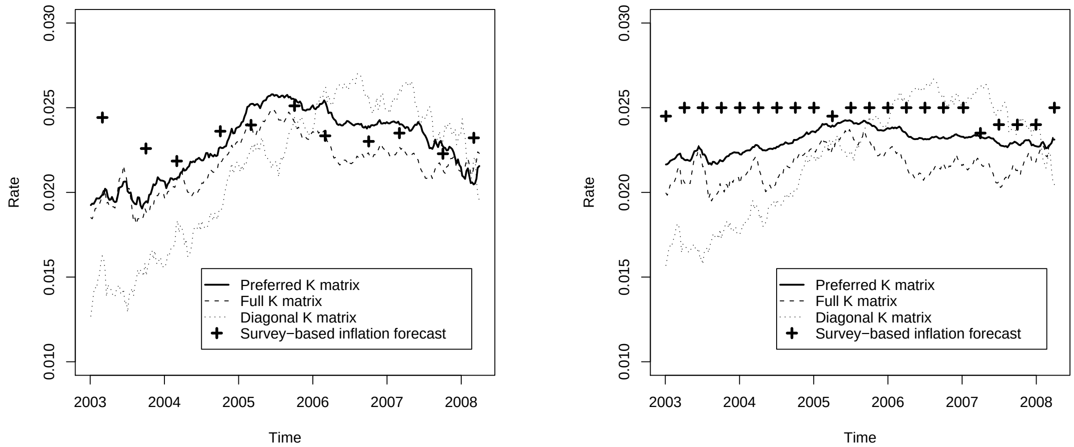
expectations.

The $K^{P,J}$ matrix, which governs factor dynamics, plays a key role in the decomposition of BEI rates. The dependence of model-implied inflation expectations on the specification of the $K^{P,J}$ matrix is illustrated in Figure 4, which shows five- and ten-year expected inflation implied by three different specifications of the joint AFNS model. The solid line is the preferred specification described earlier, and the dashed and dotted lines are based on unrestricted and diagonal $K^{P,J}$ matrices, respectively. The preferred specification, which was selected based on in-sample fit to the data, also provides about the closest match to the survey-based inflation forecasts.

Finally, for our preferred specification, we subtract each model-implied expected inflation rate from the comparable-maturity model-implied BEI rate and obtain the associated inflation risk premium (IRP). At both the five- and ten-year horizons, these premiums are fairly small, as shown in Figure 5.[14] Indeed, during our sample, these inflation premiums have varied in a range around zero of about ±50 basis points.[15]

[14]This result provides some support for the argument that the gain to the U.S. Treasury from issuing TIPS bonds instead of nominal bonds may be quite limited, as argued in Sack and Elsasser (2004).

[15]Again, in theory, the sign of the inflation risk premium depends on the covariance between the real stochastic discount factor and inflation, but there are real-world considerations as well. For example, a liquidity premium for holding TIPS instead of nominal Treasury bonds would show up as a negative inflation risk premium.



(a) Five-year expected inflation (b) Ten-year expected inflation

Figure 4: **Expected Inflation Implied by Alternative Specifications.**
Model-implied inflation expectations at the five- and ten-year horizons are shown for our joint AFNS model with the preferred specification of the $K^{P,J}$ matrix, with an unrestricted full $K^{P,J}$ matrix, and with a diagonal $K^{P,J}$ matrix.

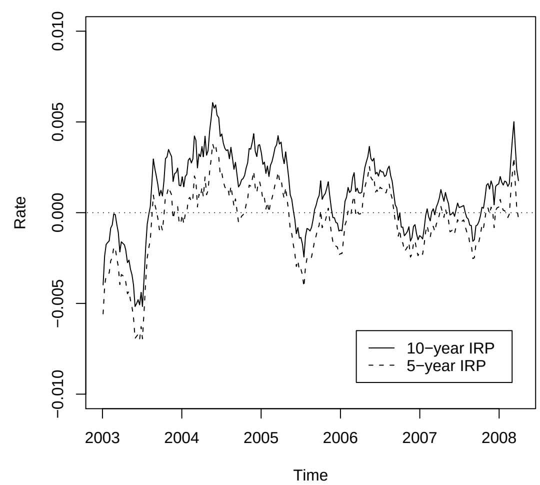


Figure 5: **Model-Implied Inflation Risk Premiums.**
The five- and ten-year inflation risk premiums (IRP) that are implied from the joint AFNS model are plotted.

5.3 Comparison to the literature

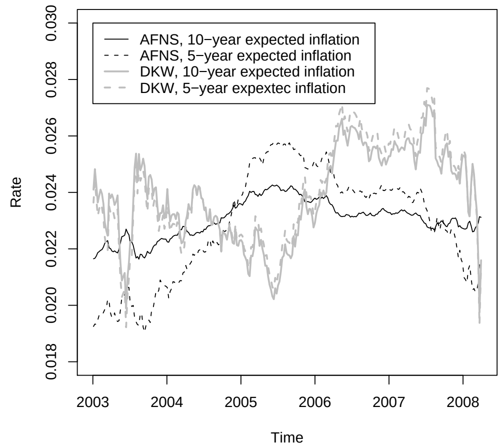
Our results can be usefully compared to the findings of three recent papers in the literature that also decompose U.S. BEI rates using empirical affine AF models of nominal and real yields. The earliest of these studies is by Chen, Liu, and Cheng (2005), who estimate a two-factor AF model using the weekly nominal and TIPS yields data from January 1998 to December 2004. Their estimated inflation risk premiums are quite stable. The 5-year premium averages about zero, similar to ours, while their 10-year premium averages around 130 basis points, which is much higher than our estimate (even for only the two years of overlap between their estimation sample for real yields and ours). However, it seems likely that their model estimates are inappropriately influenced by the use of the TIPS yields data from 1998 through 2002, when the market exhibited little volume and poor liquidity.

As in our analysis, the two other recent U.S. studies also discard these earlier readings on TIPS yields. Specifically, Hördahl and Tristani (2008) decompose BEI rates using an AF macro-finance model that incorporates monthly data on nominal yields, real yields (since 2003), inflation, the output gap, and survey forecasts for inflation and the three-month interest rate. Similar to our results, they find that the ten-year inflation risk premium over the past several years fluctuates within a band of ± 50 basis points around zero. Of course, one of the distinguishing features of our analysis is that it uses a "yields-only" specification, which can provide a market-based reading of inflation expectations that is separate from survey readings or inflation data.

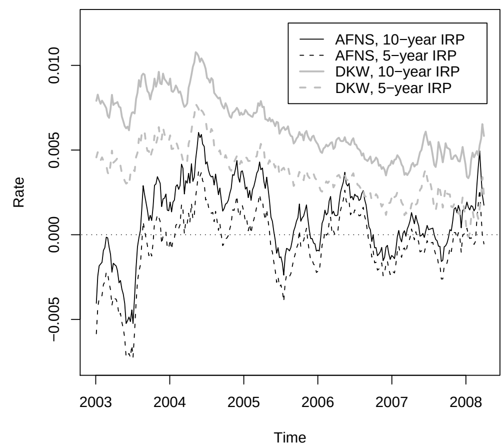
Finally, DKW also decompose BEI rates with an affine AF model. Their estimation uses weekly data from the nominal Treasury yield curve, weekly data from the real TIPS yield curve (since 2005), monthly data on inflation, and monthly survey forecasts of short-term nominal interest rates and inflation. Figure 6(a) compares their estimated five- and ten-year inflation expectations (based on their preferred model that includes TIPS yields to our AFNS results.)[16] The average values for their five- and ten-year inflation expectation measures over our sample period of January 2003 through March 2008 are 2.40 percent and 2.39 percent, respectively, which are similar to our values of 2.28 percent and 2.30 percent, respectively. Note, however, that their five- and ten-year inflation expectation measures have almost identical dynamics–in contrast to our results. Their results likely reflect the use of a single factor to capture the levels of both nominal and real Treasury yields, while our model uses separate level factors.

In addition, the dynamics of the two sets of measures are quite different; the correlation coefficients between the DKW and AFNS measures are 0.14 for the five-year horizon and -0.27 for the ten-year horizon. However, the DKW inflation measures do not match the survey measures of inflation expectations very well, even though their models include both inflation

[16] We thank Min Wei for sharing up-to-date, high frequency results with us.



(a) Model-implied inflation expectations.



(b) Model-implied IRP.

Figure 6: **Model-Implied Inflation Expectations and Risk Premiums.**
Model-implied inflation expectations and inflation risk premiums (IRP) at the five-and ten-year horizons are shown for our joint AFNS model and the DKW model.

and survey data. The correlation between the DKW five-year and survey measures of five-year inflation expectations is -0.08, while our the AFNS and survey measures have a correlation of 0.40. We can also compare our inflation risk premiums to the DKW estimates, as shown in Figure 6(b). The unconditional means for the AFNS five- and ten-year inflation risk premiums measures are both about -5 basis points, while for the DKW measures, these means are 36 and 64 basis points, respectively. In addition to these differences in unconditional moments, the correlation coefficients between the AFNS and DKW measures of inflation risk premiums are relatively low at 0.25 and 0.38, at the five- and ten-year horizons, respectively.

6 Conclusion

This paper estimates an arbitrage-free model with four latent factors that can capture the dynamics of both the nominal and real Treasury yield curves well and can decompose BEI rates into inflation expectations and inflation risk premiums. The model-implied measures of inflation expectations are correlated closely with survey measures, while the estimated inflation risk premiums fluctuate in fairly close range around zero. The empirical results suggest that long-term inflation expectations have been well-anchored in the period from year-end 2002 through the first quarter of 2008.

Our proposed model has a distinct advantage in that it can be easily estimated because it adopts the dynamic arbitrage-free Nelson-Siegel structure developed by CDR.[17] Such easy and robust estimation, implemented with the Kalman filter, enables quick updating of the model to incorporate new observations and facilitates the monitoring and forecasting of Treasury yield curves on a real-time basis. The resulting high-frequency measures should be quite desirable to policymakers, central bank staff, and financial market practitioners.

[17]Christensen, Lopez, and Rudebusch (2008) provide another application of the AFNS structure that also demonstrates its favorable estimation properties.

Appendix

Model estimation procedures

We estimate all models using the Kalman filter; see Harvey (1989) for further details. The measurement equation for the bond yields is given by

$$y_t = A + BX_t + \varepsilon_t,$$

where ε_t represents measurement errors that are assumed to be independently and identically distributed (i.i.d.) for each maturity included in the data sample.

For continuous-time Gaussian models, the conditional mean vector and covariance matrix are given by

$$\begin{aligned} E^P[X_T|\mathcal{F}_t] &= (I - \exp(-K^P\Delta t))\mu^P + \exp(-K^P\Delta t)X_t, \\ V^P[X_T|\mathcal{F}_t] &= \int_0^{\Delta t} e^{-K^P s}\Sigma\Sigma'^{-(K^P)'s} ds, \end{aligned}$$

where $\Delta t = T - t$ and $\exp(-K^P\Delta t_i)$ is a matrix exponential.

Stationarity of the system under the P-measure is ensured if the real component of all the eigenvalues of K^P is positive, and this condition is imposed in all estimations. For this reason, we can start the Kalman filter at the unconditional mean and covariance matrix,[18] denoted as

$$\widehat{X}_0 = \mu^P \quad \text{and} \quad \widehat{\Sigma}_0 = \int_0^{\infty} e^{-K^P s}\Sigma\Sigma'^{-(K^P)'s} ds.$$

The state equation in the Kalman filter is given by

$$X_{t_i} = \Phi^0_{\Delta t_i} + \Phi^1_{\Delta t_i} X_{t_{i-1}} + \eta_{t_i},$$

where

$$\Phi^0_{\Delta t_i} = (I - \exp(-K^P\Delta t_i))\mu^P, \quad \Phi^1_{\Delta t_i} = \exp(-K^P\Delta t_i), \quad \text{and} \quad \eta_{t_i} \sim N\Big(0, \int_0^{\Delta t_i} e^{-K^P s}\Sigma\Sigma'^{-(K^P)'s} ds\Big)$$

with $\Delta t_i = t_i - t_{i-1}$.

In the Kalman filter estimations, all measurement errors are assumed to be i.i.d. white

[18] In the estimation, $\int_0^\infty e^{-K^P s}\Sigma\Sigma'^{-(K^P)'s} ds$ is approximated by $\int_0^{10} e^{-K^P s}\Sigma\Sigma'^{-(K^P)'s} ds$.

noise. Thus, the error structure is in general given by

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right].$$

In the estimation, each maturity of the Treasury bond yields has its own measurement error standard deviation, $\sigma^2(\tau_i)$.

The linear least-squares optimality of the Kalman filter requires that the white noise transition and measurement errors be orthogonal to the initial state; i.e.,

$$E[f_0\eta_t'] = 0, \quad E[f_0\varepsilon_t'] = 0.$$

Finally, the standard deviations of the estimated parameters are calculated as

$$\Sigma(\widehat{\psi}) = \frac{1}{T}\Big[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \log l_t(\widehat{\psi})}{\partial \psi}\frac{\partial \log l_t(\widehat{\psi})}{\partial \psi}'\Big]^{-1},$$

where $\widehat{\psi}$ denotes the optimal parameter set.

A three-factor joint model for nominal and real yields

We considered an alternative three-factor model of nominal and real yields. The instantaneous nominal risk-free rate was given by

$$r_t^N = L_t^N + S_t^N,$$

and the usual AFNS dynamics of the three state variables under the pricing measure were imposed. Given that only two factors were needed to model the variation in the real TIPS yields and that both of these factors were correlated with the corresponding nominal yield risk factors, a reasonable specification of the instantaneous real yield process was

$$r_t^R = \alpha_L^R L_t^N + \alpha_S^R S_t^N.$$

Thus, this instantaneous real yield was driven by the same two factors that drive the nominal short rate process.

Unfortunately, the estimated three-factor model (with a full $K^{P,N}$ matrix) performs relatively poorly. Table 8 reports the fitted errors for the eight nominal yield maturities and the six real yield maturities. This table shows that the three-factor model has a significantly

Maturity in months	Three-Factor Model	
Nom. yields	Mean	RMSE
3	-4.59	43.54
6	-4.58	30.46
12	-2.41	15.24
24	0.63	2.74
36	0.80	2.91
60	-0.56	2.33
84	-0.31	0.99
120	0.00	0.00
TIPS yields	Mean	RMSE
60	-24.91	28.34
72	-15.42	18.25
84	-8.51	10.56
96	-3.53	4.63
108	0.00	0.00
120	2.43	3.61

Table 8: **Summary Statistics for Three-Factor Joint AFNS Models.**
The mean and root mean squared error of the fitted errors for the three-factor joint AFNS model. All numbers are measured in basis points. The nominal yields cover the period from January 6, 1995, to March 28, 2008, while the real TIPS yields cover the period from January 3, 2003, to March 28, 2008.

deteriorated fit for the three-month, six-month, and one-year nominal yields, and for the five-year, six-year, and seven-year real TIPS yields. Therefore, we adopted the joint four-factor model described in the text.

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