Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.

Bank regulation under fire sale externalities

Gazi I. Kara and S. Mehmet Ozsoy

2016-026

Please cite this paper as:
Kara, Gazi Ishak, and S. Mehmet Ozsoy (2016). "Bank regulation under fire sale externalities," Finance and Economics Discussion Series 2016-026. Washington: Board of Governors of the Federal Reserve System, http://dx.doi.org/10.17016/FEDS.2016.026.

Bank regulation under fire sale externalities*

Gazi Ishak Kara[†]
S. Mehmet Ozsoy[‡]

March, 2016

Abstract

This paper examines the optimal design of and interaction between capital and liquidity regulations in a model characterized by fire sale externalities. In the model, banks can insure against potential liquidity shocks by hoarding sufficient precautionary liquid assets. However, it is never optimal to fully insure, so realized liquidity shocks trigger an asset fire sale. Banks, not internalizing the fire sale externality, overinvest in the risky asset and underinvest in the liquid asset in the unregulated competitive equilibrium. Capital requirements can lead to less severe fire sales by addressing the inefficiency and reducing risky assets—however, we show that banks respond to stricter capital requirements by decreasing their liquidity ratios. Anticipating this response, the regulator preemptively sets capital ratios at high levels. Ultimately, this interplay between banks and the regulator leads to inefficiently low levels of risky assets and liquidity. Macroprudential liquidity requirements that complement capital regulations, as in Basel III, restore constrained efficiency, improve financial stability and allow for a higher level of investment in risky assets.

Keywords: Bank capital regulation, liquidity regulation, fire sale externality, Basel III

JEL Codes: G20, G21, G28.

---

*We are grateful to Guido Lorenzoni, Harald Uhlig, William Bassett and seminar participants at the Federal Reserve Board of Governors, Ozyegin University, International Monetary Fund, Federal Reserve Bank of Atlanta, AEA Annual Meeting in Boston, Financial Intermediation Research Society Conference in Reykjavik, Midwest Finance Conference in Chicago, Midwest Macro Conference in Miami, FMA Annual Meeting in Orlando, Effective Macro-Prudential Instruments Conference at the University of Nottingham, Turkish Finance Workshop at Bilkent University, Istanbul Technical University, Tenth Seminar on Risk, Financial Stability and Banking of the Banco Central do Brasil, and Sixth Annual Financial Market Liquidity Conference in Budapest for helpful comments and suggestions. All errors are ours. The analysis and the conclusions set forth are those of the authors and do not indicate concurrence by other members of the research staff or the Board of Governors of the Federal Reserve.

[†]Contact author: Office of Financial Stability Policy and Research, Board of Governors of the Federal Reserve System. 20th Street and Constitution Avenue NW, Washington, D.C. 20551. Email: Gazi.I.Kara@frb.gov

[‡]Ozyegin University, Faculty of Business, Nisantepe Mah., Orman Sok., 34794 Istanbul, Turkey. Email: mehmet.ozsoy@ozyegin.edu.tr

1 Introduction

The recent financial crisis led to a redesign of bank regulations, with an emphasis on the macroprudential aspects of regulation. Prior to the crisis, capital adequacy requirements were the dominant tool of bank regulators around the world. The crisis, however, revealed that even well-capitalized banks can experience a deterioration of their capital ratios due in part to illiquid positions. Several financial institutions faced liquidity constraints simultaneously, which created an urgent need for regulators and central banks to intervene in markets to restore financial stability. Without the unprecedented liquidity and asset price supports of leading central banks, those liquidity problems could have resulted in a dramatic collapse of the financial system. The experience brought liquidity and its regulation into the spotlight.[1] A third generation of bank regulation principles, popularly known as Basel III, strengthens the previous Basel capital adequacy accords by adding macroprudential aspects and liquidity requirements such as the liquidity coverage ratio (LCR) and net stable funding ratio.

Several countries, including the United States and the countries in the European Union, have already adopted Basel III liquidity requirements together with the enhanced capital requirements. However, the guidance from theoretical literature on the regulation of liquidity and the interaction between liquidity and capital regulations is quite limited, as emphasized by Bouwman (2012) as well. The scarcity of academic guidance is also apparent in a 2011 survey paper on illiquidity by Jean Tirole, in which he succinctly asks, "Can we trust the institutions to properly manage their liquidity, once excessive risk taking has been controlled by the capital requirement?" (Tirole, 2011).

In this paper, we show that banks' choices of capital and liquidity ratios in an unregulated competitive equilibrium are inefficient under a fire sale externality and we investigate the optimal design of capital and liquidity regulations to restore the constrained efficiency. In particular, we analyze whether it suffices to introduce capital regulations alone and let banks freely choose their liquidity ratios or whether liquidity also needs to be regulated. We consider a three-period model in which a continuum of banks have access to two types of assets. Banks have to decide at the initial period how many risky and liquid assets to carry in their portfolio. We allow for a flexible balance sheet size, such that banks can increase both their risky and liquid assets at the same time. Banks start with a fixed amount of equity capital and borrow the funds necessary to finance their portfolio from consumers.

The risky asset has a constant return but requires, with a known probability, additional investment in the future before collecting returns. This additional investment cost creates a liquidity need, which is proportional to the amount of risky assets on a bank's balance sheet. The liquid asset provides zero net return; however, it can be used to cover the additional investment cost. A limited-commitment problem prevents banks from raising additional external finance in the second

[1]See Rochet (2008), Bouwman (2012), Stein (2013), Tarullo (2014) and Allen (2014) for recent discussions on the regulation of bank liquidity.

period. Therefore, if liquidity from the initial period is not enough to offset the shock, the only other option is for the banks to sell some of their risky assets to outside investors to save the remaining risky assets.[2] This sell-off of risky assets takes the form of fire sales because outside investors' demand for risky assets is downward-sloping: Outside investors are less productive in managing the risky asset, and the marginal product of each risky asset decreases as the amount of risky assets managed by outside investors increases. Thus, outside investors offer a lower price when banks try to sell a higher quantity of risky assets. A lower price, in turn, requires each bank to further increase the quantity of risky assets to be sold, creating an externality that goes through asset prices.

Atomistic banks do not take into account the effect of their initial portfolio choices on the fire sale price. If banks hold more risky assets, the liquidity need in case of an aggregate shock is greater. As a result, there are more fire sales and a lower fire sale price, which in turn requires each bank to sell more risky assets to raise the required liquidity. Similarly, smaller liquidity buffers in the banks' initial portfolios lead to greater fire sales and a lower fire sale price. We compare the unregulated competitive equilibrium in which banks freely choose their capital and liquidity ratios to the allocations of a constrained planner. Without internalizing the effect on the fire sale price, banks overinvest in the risky asset and underinvest in the liquid assets in the unregulated competitive equilibrium. The constrained planner, in contrast, is subject to the same constraints as the private agents but internalizes the effect of initial allocations on the fire sale price. We also investigate how the constrained efficient allocations can be implemented using quantity-based capital and liquidity regulations, as in the Basel Accords.

The constrained inefficiency of competitive equilibrium in this paper is due to the existence of a pecuniary externality under incomplete markets. In our framework, this is the only externality.[3] The Pareto suboptimality due to pecuniary externalities is well known in the literature. Greenwald and Stiglitz (1986), for instance, show that pecuniary externalities by themselves are not a source of inefficiency but can lead to significant welfare losses when markets are incomplete or when there is imperfect information. More recently, Lorenzoni (2008) shows that the combination of pecuniary externalities in the fire sale market and limited commitment in financial contracts leads to too much investment in risky assets in the competitive equilibrium.

In this paper, the incompleteness of markets arises from the financial constraints of bankers in the interim period. Specifically, similarly to Kiyotaki and Moore (1997), Lorenzoni (2008), Korinek (2011), and Stein (2012) we assume that a limited-commitment problem prevents banks from borrowing the funds necessary for restructuring when the liquidity shock hits. If the markets are complete and banks can borrow by pledging the future return stream from the assets, fire sales

[2] The liquidity shock is aggregate in nature; therefore, the liquidity need cannot be satisfied within the banking system, as all the banks are in need of liquidity. This assumption is not crucial for the results. In Section 5.4, we study the case with idiosyncratic shocks.

[3] We do not model agency or information problems that the literature has traditionally used to justify capital or other bank regulations.

are avoided. In this first-best world, there is no need for either capital or liquidity requirements because a systemic externality in the financial markets no longer exists.

Although the probability and size of a liquidity shock are exogenous in the model, whether fire sales take place in equilibrium is endogenously determined, as is the amount of fire sales. In principle, banks can perfectly insure themselves against fire sale risk by holding sufficiently high liquidity. However, such insurance is never optimal. The intuition is straightforward. The marginal return on liquid assets is greater than one as long as there are fire sales. Perfect insurance guarantees that no fire sale takes place, and as a result the marginal return on liquid assets is equal to one, which is dominated by the marginal return on risky assets. In other words, there is no need to hoard any liquidity when there is no fire sale risk. Thus, banks' optimal choice of liquidity is less than the amount sufficient to avoid fire sales completely: In equilibrium, fire sales take place when the liquidity shock hits.

Our results indicate that the constrained efficient allocation can be achieved with joint implementation of capital and liquidity regulations (*complete regulation*). In particular, a regulator can implement the optimal allocations by imposing a minimum risk-weighted capital ratio and a minimum liquidity ratio as a fraction of risky assets. The regulation required is macroprudential because it addresses the instability in the banking system by targeting aggregate capital and liquidity ratios. Banks hold liquid assets for microprudential reasons even if there is no regulation on liquidity because they can use these resources to protect against liquidity shocks. Liquidity is advantageous from a macroprudential standpoint as well: Higher liquidity holdings lead to less-severe decreases in asset prices during times of distress. However, banks fail to internalize this macroprudential aspect of liquidity, which results in inefficiently low liquidity ratios when there is no regulation. Similarly, banks neglect the macroprudential effects of capital ratios and end up choosing inefficiently low capital ratios in the competitive equilibrium. A minimum risk-weighted capital ratio requirement combined with a minimum liquidity ratio, as in Basel III, can restore constrained efficiency.

We then use this model to answer Tirole's question, mentioned above, by studying a regulatory framework with capital requirements alone, similar to the pre-Basel III episode, which we call *partial regulation*. In this setup, banks respond to the introduction of capital regulations by decreasing their liquidity ratios further below the already inefficient levels in the competitive equilibrium. If there is no regulation, banks choose a composition of risky and safe assets in their portfolio that reflects their privately optimal level of risk-taking. When the level of risky investment is limited by capital regulations, banks reduce the liquidity of their portfolio in order to get closer to their privately optimal level of fire sale risk. This is, in a sense, an unintended consequence of capital regulation: *Capital regulation improves financial stability by limiting aggregate risky investment, which in turn weakens banks' incentives to hold liquidity because the marginal benefit of liquidity decreases with financial stability.* The regulator tightens capital regulations under a capital ratio regime to offset banks' lower liquidity ratios, reducing socially profitable long-term investments. As a result, bank capital ratios under partial regulation are inefficiently high.

The aforementioned findings have important policy implications. The lack of complementary liquidity requirements leads to inefficiently low levels of long-term investments and severe financial crises, undermining the purpose of capital adequacy requirements. Our results indicate that the pre-Basel III regulatory framework, with its focus on capital requirements, was inefficient and ineffective in addressing systemic instability caused by liquidity shocks, and that Basel III liquidity regulations are a step in the right direction.

Our contribution is threefold. First, to the best of our knowledge, this is one of the first papers to study the interaction between capital and liquidity regulations. We show that capital regulation alone cannot restore constrained efficiency and that augmenting capital regulation with liquidity regulation both restores constrained efficiency and improves financial stability. Second, we contribute to the fire sales literature by introducing an explicit role for safe assets and showing that even though banks can perfectly hedge against fire sale risk by holding sufficient liquidity, they still choose to take some of this risk. Moreover, even the constrained planner, while choosing a higher liquidity ratio than unregulated banks, takes some fire sale risk. Third, the paper contributes to the theory of economic policy making. We show that even though there is only one fire sale externality, capital or liquidity regulation alone is not enough to achieve the socially optimal level of fire sales. This result complements the Tinbergen rule, which argues that the number of policy tools must be at least as high as the number of policy objectives. The target of the regulator in this model is to maximize the expected welfare. The regulator cannot reach this target by using capital or liquidity regulations alone because there are two distorted margins in the unregulated competitive equilibrium resulting from two independent choice variables of banks: capital and liquidity ratios. Both of these choices affect the amount of fire sales and the price of assets in equilibrium, but these effects are not internalized by atomistic banks. Therefore, achieving the social optimum always requires using both capital and liquidity regulatory tools.

The paper proceeds as follows. Section 2 contains a brief summary of related literature. Section 3 provides the basics of the model and presents the unregulated competitive equilibrium and the constrained planner's problem. Section 4 compares two alternative regulatory frameworks: complete regulation (both capital and liquidity regulations) and partial regulation (only capital regulation). Section 5 investigates the robustness of the results to some changes in the model environment. Section 6 concludes. The appendix contains the closed-form solutions of the model and proofs.

2 Literature review

Even though capital regulations have been studied extensively on their own, we are aware of only a few papers that investigate the interaction between capital and liquidity regulations and their optimal determination. Kashyap, Tsomocos, and Vardoulakis (2014) consider an extended version of the Diamond and Dybvig (1983) model to investigate the effectiveness of several bank regulations in addressing two common financial system externalities:[4] excessive risk-taking due to limited liability and bank runs. The central message of the paper is that a single regulation alone is never sufficient to correct for the inefficiencies created by these two externalities. Unlike our paper, their paper does not consider fire sale externalities, which causes a divergence in our results. For example, in their paper, optimal regulation does not necessarily involve capital or liquidity regulations.

Walther (2015) also studies macroprudential regulation in a model characterized by pecuniary externalities due to fire sales. In his setup, the fire sale price is exogenously fixed and the socially optimal outcome is to have "no fire sales" in equilibrium, whereas in our paper partial fire sales are not only allowed, they are also optimal. Walther shows that both macroprudential regulation and Pigouvian taxation can achieve the "no fire sales" outcome; however, implementation of Pigouvian taxation requires more information. Pigouvian taxation serves as an important theoretical benchmark, yet it is not part of the toolkit designed by the Basel Committee. Our paper analyzes quantity-based regulations, as in the Basel Accords.

De Nicoló, Gamba, and Lucchetta (2012) consider a dynamic model of bank regulation and shows that liquidity requirements, when added to capital requirements, eliminate the benefits of mild capital requirements by hampering bank maturity transformation and, hence, result in lower bank lending, efficiency, and social welfare. In that model, liquidity is only welfare-reducing because, unlike our paper, the authors do not consider the role of liquidity in insuring banks against the fire sale risk.

Covas and Driscoll (2014) study the introduction of liquidity requirements on top of existing capital requirements in a nonlinear dynamic general equilibrium model. They show that the presence of liquidity regulation makes bank loans less sensitive to the capital ratio and that the quantitative macroeconomic impacts of these regulatory tools are larger in partial equilibrium. Unlike Covas and Driscoll (2014) we study the socially optimal outcome and how to reach it using capital and liquidity regulation.

Even though the literature on the interaction between capital and liquidity requirements is limited, there are studies that examine the interaction between different tools available to regulators. Acharya, Mehran, and Thakor (2015) show that the optimal capital regulation requires a two-tiered capital requirement with some bank capital invested in safe assets. The special capital should be unavailable to creditors upon failure so as to retain market discipline and should be available to shareholders only contingently on good performance in order to contain risk-taking.

Arseneau et al. (2015) study the interaction between secondary market liquidity and firms' capital structure when search frictions in the secondary market generate a liquidity premium in the primary market. Agents do not internalize the effects of portfolio allocations in the primary

[4] The authors consider the following regulations: deposit insurance, loan-to-value limits, dividend taxes, and capital and liquidity ratio requirements.

market on the secondary market illiquidity, and thus on the liquidity premium. The unregulated equilibrium is constrained inefficient and the authors, focusing on quantitative easing, show that, similar to our result, two policy tools (both asset purchases and interest on reserves) are needed to restore the constrained efficiency.

Hellmann, Murdock, and Stiglitz (2000) show that while capital requirements can induce prudent behavior, they lead to Pareto-inefficient outcomes by reducing banks' franchise values, hence providing incentives for gambling. Pareto-efficient outcomes can be achieved by adding deposit-rate controls as a regulatory instrument. Such controls restore prudent behavior by increasing franchise values. Similar to their result, we show that capital requirements provide Pareto efficiency only if they are combined with liquidity requirements.

As in our paper, a few seminal papers have pointed out the inefficiency of liquidity choice of banks in laissez-faire equilibrium under market incompleteness or informational frictions. Bhattacharya and Gale (1987) consider an extended version of Diamond and Dybvig (1983) with several banks and show that when banks face privately observed liquidity shocks, they underinvest in liquid assets and free-ride on the common pool of liquidity in the interbank market. Allen and Gale (2004b) show that when markets for hedging liquidity risk is incomplete, private liquidity hoardings of banks is inefficient. Whether there is too much or too little liquid assets in the laissez-faire equilibrium depends on the coefficient of relative risk aversion: if it is greater than one, the liquidity is inefficiently low.

Several papers study liquidity and its regulation without explicitly analyzing its interaction with capital requirements or its role in addressing fire sale externalities. Calomiris, Heider, and Hoerova (2013) argue that the role of liquidity requirements should be conceived not only as an insurance policy that addresses the liquidity risks in distressed times, as proposed by Basel III, but also as a prudential regulatory tool that makes crises less likely. Repullo (2005) shows, in direct contrast to our result, that a higher capital requirement reduces the attractiveness of risky investment, and hence, causes a bank to increase its investment in safe assets. In his model, the balance sheet size of bank is exogenously fixed, and hence, a decrease in risky investment necessarily implies an increase in safe assets. In contrast, we consider a model with a flexible bank balance sheet in which capital requirement decreases risky investment level, and banks respond by decreasing their liquidity ratios. Perotti and Suarez (2011) show that banks choose an excessive amount of short term debt in the presence of systemic externalities and analyze the effectiveness of liquidity regulations as in Basel III as opposed to Pigovian taxation in implementing the social optimal level of short term funding.

Farhi et al. (2009) consider a Diamond-Dybvig model with unobservable liquidity shocks and unobservable trades. They show that competitive equilibria are inefficient even if the markets for aggregate risk are complete and that optimal allocations can be implemented through a simple liquidity ratio requirement on financial intermediaries.

Our paper is also related to the literature that features financial amplification and asset fire sales, which includes the seminal contributions of Fisher (1933), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Krishnamurthy (2003, 2010), and Brunnermeier and Pedersen (2009). In our model, fire sales result from the combined effects of asset-specificity and correlated shocks that hit an entire industry or economy. This idea, originating with Williamson (1988) and Shleifer and Vishny (1992), is employed by fire sale models such as Lorenzoni (2008), Korinek (2011), and Kara (2015). These papers show that under pecuniary externalities arising from asset fire sales, there exists overinvestment in risky assets in a competitive setting compared with the socially optimal solution. However, unlike our paper, none of these papers give an explicit role for safe assets, which banks can use to completely insure themselves against the fire sale risk.

Similar to our result, Stein (2012) shows that both the liquidity and investment decision of individual banks are distorted: Banks, not internalizing the fire sale externality, rely too much on short term debt, a cheap form of financing, which in turn supports greater lending. The liquidity choice in Stein's model is on the liability side of banks' balance sheet. We model the liquidity hoarding decision on the asset side. More importantly, in Stein's setup once the liquidity choice of banks is aligned with the socially optimal level by regulation, the investment decision is also aligned automatically. Similarly, when banks are exposed to the social cost of short term financing, through Pigouvain taxation for example, marginal cost increases which brings down the bank lending to the socially optimal level. This is contrary to our results. In our paper, regulating liquidity alone or imposing a tax on it is not sufficient to guarantee the socially optimal level of investment. Both the amount of total liquidity and total investment determine the amount of fire sales, and thus should be regulated.

The constrained inefficiency of competitive markets in this paper is due to the existence of pecuniary externalities under incomplete markets. The Pareto suboptimality of competitive markets when the markets are incomplete goes back at least to the work of Borch (1962). The idea was further developed in the seminal papers of Hart (1975), Stiglitz (1982), and Geanakoplos and Polemarchakis (1986), among others. Greenwald and Stiglitz (1986) extended the analysis by showing that, in general, pecuniary externalities by themselves are not a source of inefficiency but can lead to significant welfare losses when markets are incomplete or there is imperfect information.

In our model, limited commitment problem prevents the laissez-faire markets from attaining Pareto optimality by distorting every choice variable. Therefore, banks' private choices of capital and liquidity are inefficient and reaching the second-best requires intervention in both choices of banks. This result is in the spirit of Lipsey and Lancaster (1956) who show that failure to satisfy a single Pareto condition requires distorting potentially all the other Pareto conditions in order to attain the second-best outcome.

3 Model

The model consists of three periods, $t = 0, 1, 2$; along with a continuum of banks and a continuum of consumers, each with a unit mass. There is also a unit mass of outside investors. All agents are risk-neutral and derive utility from consumption in the initial and final periods.

There are two types of goods in this economy, a consumption good and an investment good (that is, the liquid and the illiquid asset). Consumers are endowed with e units of consumption goods at $t = 0$ but none at $t = 1$ and $t = 2$.[5] Banks have a technology that converts consumption goods into investment goods one-to-one at $t = 0$. Investment goods that are managed by a bank until the last period will yield $R > 1$ consumption goods per unit. However, investment goods are subject to a liquidity shock at $t = 1$, which we discuss in detail below, and hence we refer to them as the risky assets. Risky assets can be thought as mortgage-backed securities or a portfolio of loans to firms in the corporate sector.[6] Investment goods can never be converted back into the consumption goods, and they fully depreciate after the return is collected at $t = 2$.

Banks choose at $t = 0$ how many risky assets to hold, denoted by n_i, and how many liquid (safe) assets, denoted by b_i, to put aside for each unit of risky assets. The total amount of liquid assets held by each bank is then $n_i b_i$, and b_i can be interpreted as a liquidity ratio. The return on the liquid asset is normalized to one. Therefore, the total asset size of a bank is $n_i + n_i b_i = (1+b_i)n_i$. On the liability side, each bank is endowed with E units equity capital at $t = 0$ in terms of consumption goods. The fixed amount of equity capital assumption captures the fact that it is difficult for banks to raise equity in the short-term, and it is also imposed by others in the banking literature (see for example, Almazan, 2002; Repullo, 2005; Dell'Ariccia and Marquez, 2006). Hence, each bank raises $L_i = (1 + b_i)n_i - E$ units of consumption goods from consumers at $t = 0$ to finance its portfolio of safe and risky assets.

We assume that the initial equity of banks is sufficiently large to avoid default in the bad state in equilibrium. As a result, the deposits are safe, and hence consumers inelastically supply deposits to banks at net zero interest rate at the initial period. This assumption also allows us to focus on only one friction—that is, fire sale externalities—and to study the implications of this friction for the optimal regulation of bank capital and liquidity. However, as we show in Section 5, our results are robust to relaxing this assumption and allowing bank default in equilibrium.

We assume that there is a nonpecuniary cost of operating a bank, captured by $\Phi((1 + b_i)n_i)$. The operational cost is increasing in the size of the balance sheet, $\Phi'(\cdot) > 0$, and it is convex, $\Phi''(\cdot) > 0$. This assumption, similar to the ones imposed by Van den Heuvel (2008) and Acharya (2003, 2009), ensures that the banks' problem is well defined and that there is an interior solution

[5] We assume that the initial endowment of consumers is sufficiently large, and it is not a binding constraint in equilibrium.

[6] To simplify the exposition, we abstract from modeling the relationship between banks and firms. Instead, we assume that banks directly invest in physical projects. This assumption is equivalent to assuming that there are no contracting frictions between banks and firms, as more broadly discussed by Stein (2012).

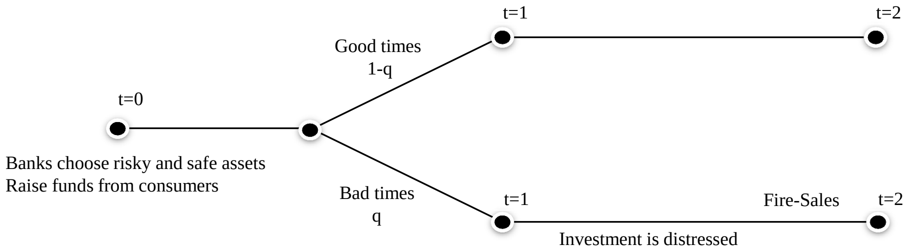
to this problem. The convex operational cost assumption allows us to have banks with flexible balance sheet size in the model. If the balance sheet size of the bank is fixed, and liquid and risky (illiquid) assets are the only assets a bank can buy, then the choice between liquid and illiquid (risky) asset boils down to a single choice—namely, an allocation problem. If a bank increases its risky assets, the amount of liquid assets in the bank's portfolio necessarily decreases because now there are fewer resources available for the liquid assets. In our framework with flexible balance sheets, banks can increase or decrease the amount of risky and liquid assets simultaneously, if it is optimal for them to do so. As a result, this setup allows us to study two independent choices of banks, as well as their interaction.

Investment and deposit collection decisions are made at time $t = 0$. The only uncertainty in the model is about the risky asset and is resolved at the beginning of $t = 1$: The economy lands in good times with probability $1 - q$ and in bad times with probability q. In good times, no bank is hit with liquidity shocks, and therefore no further action is taken. Banks keep managing their investment goods and in the final period realize a total return of $Rn_i + n_i b_i$. However, in bad times, the risky assets are distressed. In case of distress, the investment (risky assets) has to be restructured in order to remain productive. Restructuring costs are equal to $c \leq 1$ units of consumption goods per unit of the risky asset. If c is not paid, the risky investment is scrapped (that is, it fully depreciates). For the case of bank loans, the liquidity shock can be considered a utilization of committed credit lines or loan commitments, which increases in bad times (Holmström and Tirole, 2001; Stein, 2013). Firms may need the extra resources to cover operating expenses or other cash needs. For mortgage-backed securities, a liquidity shock may arise if investors' risk perception of these assets changes in bad times and requires banks to post extra margin in order to keep financing the investment.

A bank can use the liquid assets hoarded from the initial period, $n_i b_i$, to carry out the restructuring of the distressed investment at $t = 1$. However, if the liquid assets are not sufficient to cover the entire cost of restructuring, the bank needs external finance. Other than banks, only outside investors are endowed with liquid resources at this point. Because of a limited-commitment problem, banks cannot borrow the required resources from outside investors. In particular, similarly to Kiyotaki and Moore (1997) and Korinek (2011), we assume that banks can only pledge the market value, not the dividend income, of their asset holdings next period to outside investors. This assumption prevents banks from borrowing between the interim and the final periods because the value of all assets are zero in the final period, and hence banks have no collateral to pledge to outside investors in the interim period.[7] In other words, this assumption states that the contracts

[7] For simplicity, we assume that the commitment problem is extreme (that is, banks cannot commit to pay any fraction of their production to outside investors). Assuming a milder but sufficiently strong commitment problem where banks can commit a small fraction of their production, as Lorenzoni (2008) and Gai et al. (2008) do, does not change the results of this paper. If we complete the markets by allowing banks to borrow from outside investors by pledging the all-future-return stream from the assets, there would not be a reason for fire sales and the first-best world would be established. In the first-best world, there would not be a need for regulation as the pecuniary externality

Figure 1: Timing of the model



between banks and outside investors are not enforceable.

The only way for banks to raise the funds necessary for restructuring is by selling some fraction of the risky asset to outside investors in an exchange of consumption goods.[8] Allen and Gale (2004b) in part build a model arguing that in the realm of financial intermediaries, markets for hedging liquidity risk are likely to be more incomplete than the markets for hedging asset return risk. We take the same approach here. Our assumption that the return of risky assets is nonstochastic essentially captures the efficient sharing of that risk and admits that asset returns are not the source of price volatility. If markets for hedging liquidity risk were complete as well, there would be no need to sell assets to obtain liquidity (Allen and Gale, 2004b, 2005).

The asset sales by banks are in the form of fire sales: The risky asset is traded below its fundamental value for banks, and the price decreases as banks try to sell more assets. Banks retain only a fraction, γ, of their risky assets after fire sales, which depends on banks' liquidity shortages as well as on the fire sale price of risky asset. The sequence of events is illustrated in Figure 1.

We first solve the competitive equilibrium of the model when there is no regulation on banks. Second, we present the constrained planner's problem and analyze its implementation using both quantity-based capital and liquidity requirements as in the Basel Accords. Last, we consider a partially regulated economy in which there is capital regulation but no regulation on bank liquidity ratios. The liquidity regulation requires the banks to satisfy a minimum liquidity ratio such that $b_i \geq \underline{b}$. The capital regulation requires banks to satisfy a minimum risk-weighted capital ratio, $\underline{k}$, at $t = 0$, such that $k_i = E/n_i \geq \underline{k}$. Because the inside equity of banks, E, is fixed in our model, the minimum risk-weighted capital ratio regulation is equivalent to a regulation in the form of an upper limit on initial risky investment levels, $\bar{n}$, such that banks' investments have to satisfy $n_i \leq \bar{n}$, where $\bar{n} \equiv E/\underline{k}$. For analytical convenience, we use the upper bound on risky investment

in financial markets would be eliminated.

[8]An alternative story would be that households come in two generations, as in Korinek (2011), and the assets produce a (potentially risky) return in the interim period in addition to the safe return in the final period. In this case, banks can borrow from the first-generation households at the initial period because they have sufficient collateral to back their promises in the interim period, but banks cannot borrow from second-generation households because the value of all assets is zero in the final period. In this alternative story, second-generation households will be the buyers of assets from banks, and they will employ assets in a less productive technology to produce returns in the final period similar to outside investors here.

formulation for capital regulation in the rest of the paper.

3.1 Crisis and fire sales

The decision of agents at time t=0 depends on their expectations regarding the events at time $t = 1$. Thus, applying the solution by backwards induction, we first analyze the equilibrium at the interim period in each state of the world for a given set of investment levels. We then study the equilibrium at $t = 0$. Note that if the good state is realized at $t = 1$, banks take no further action and obtain a total return of $\pi_i^{Good} = Rn_i + b_i n_i$ at the final period, $t = 2$. Therefore, for the interim period $t = 1$, studying the equilibrium only for bad times is sufficient. We start with the problem of outside investors in bad times, then analyze the problem of banks.

3.1.1 Outside investors

Outside investors are endowed with large resources of consumption goods at $t = 1$, and they can purchase investment goods from the banks. Some examples of outside investors who are available to buy assets from the banking industry in distress times are private equity firms, hedge funds, or Warren Buffet (Diamond and Rajan, 2011). Let us denote the amount of investment goods they buy from the banks by y. The outside investors have a concave production technology and employ these investment goods to produce $F(y)$ units of consumption goods at $t = 2$. Let P denote the market price of the investment good in bad times at $t = 1$.[9] Each outside investor takes the market price as given and chooses the amount of investment goods to buy, y, in order to maximize net returns from investment at $t = 2$:

$$\max_{y \geq 0} F(y) - Py.$$

The first-order condition of the investors' maximization problem, $F'(y) = P$, determines the outside investors' (inverse) demand function for the investment good. We can define their demand function, $Q^d(P)$, as follows: $Q^d(P) \equiv F'(P)^{-1} = y$.

Assumption 1 (*Concavity*)**.** *$F'(y) > 0$ and $F''(y) < 0$ for all $y \geq 0$, with $F'(0) \leq R$.*

The Concavity assumption establishes that outside investors are less efficient than the banks. Outside investors' return is strictly increasing in the amount of assets employed, $F'(y) > 0$, and they face decreasing returns to scale in the production of consumption goods, $F''(y) < 0$, whereas banks are endowed with a constant returns to scale technology, as described earlier. Together with concavity, $F'(0) \leq R$ implies that outside investors are less productive than banks at each level of investment goods employed.

The concavity of the return function implies that the demand function of outside investors for investment goods is downward-sloping (see Figure 2). In other words, outside investors require

[9] The price of the investment good at $t = 0$ will be one as long as there is positive investment, and the price at $t = 2$ will be zero because the investment good fully depreciates at this point.

higher discounts to absorb more assets from distressed banks at $t = 1$. The decreasing returns to scale technology assumption is a reduced way of modeling the existence of industry-specific heterogeneous assets, similarly to Kiyotaki and Moore (1997), Lorenzoni (2008), and Korinek (2011). In this more general setup, outside investors would first purchase assets that are easy to manage, but as they continue to purchase more assets, they would need to buy those that require increasingly sophisticated management and operation skills.

The idea that some assets are industry-specific and, hence, less productive in the hands of outsiders has its origins in Williamson (1988) and Shleifer and Vishny (1992).[10] In these studies, the authors claim that when major players in such industries face correlated liquidity shocks and cannot raise external finance due to debt overhang, agency, or commitment problems, they have to sell assets to outsiders. Outsiders are willing to pay less than the value in best use for the assets of distressed enterprises because they do not have the specific expertise to manage these assets well and therefore face agency costs of hiring specialists to run these assets.[11] For instance, monitoring and collection skills of loan officers greatly affect the value of bank assets, particularly bank loans. The lack of such skills among outsiders creates a deadweight cost of fire sales (Acharya et al., 2011).

Empirical and anecdotal evidence suggests the existence of fire sales of physical as well as financial assets. Using a large sample of commercial aircraft transactions, Pulvino (2002) shows that distressed airlines sell aircraft at a 14 percent discount from the average market price. This discount exists when the airline industry is depressed but not when it is booming. Coval and Stafford (2007) shows that fire sales exist in equity markets when mutual funds engage in sales of similar stocks.

Next, we need to impose more structure on the return function of outside investors in order to ensure that the equilibrium of this model exists and is unique.

Assumption 2 (Elasticity)**.**

$$\epsilon^d = \frac{\partial Q^d(P)}{\partial P}\frac{P}{Q^d(P)} = \frac{F'(y)}{yF''(y)} < -1 \quad \textit{for all } y \geq 0$$

The *Elasticity* assumption states that outside investors' demand for the investment good is elastic. This assumption implies that the amount spent by outside investors on asset purchases, $Py = F'(y)y$, is strictly increasing in y. Therefore, we can also write the *Elasticity* assumption as $F'(y) + yF''(y) > 0$. If this assumption was violated, multiple levels of asset sales would raise a given amount of liquidity, and multiple equilibria in the asset market at $t = 1$ would be possible.

[10] Industry-specific assets can be physical or they can be portfolios of financial intermediaries (Gai et al., 2008).

[11] As opposed to the asset specificity idea discussed earlier, in Allen and Gale (1994, 1998) and Acharya and Yorulmazer (2008), the reason for fire sales is the limited amount of available cash in the market to buy long-term assets offered for sale by agents who need liquid resources immediately. The scarcity of liquid resources leads to necessary discounts in asset prices, a phenomenon known as "cash-in-the-market pricing." Uhlig (2010) analyzes other market failures that might result in fire sales.

This assumption is imposed by Lorenzoni (2008) and Korinek (2011) in order to rule out multiple equilibria under fire sales.[12]

Assumption 3 (*Regularity*)**.** $F'(y)F'''(y) - 2F''(y)^2 \leq 0$ *for all* $y \geq 0$.

The *Regularity* assumption holds whenever the demand function of outside investors is log-concave, but it is weaker than log-concavity.[13] Log-concavity of a demand function is a common assumption used in the Cournot games literature.[14] This assumption ensures the existence and uniqueness of an equilibrium in a simple *n-player* Cournot game. In our setup, this assumption guarantees that the objective functions are well behaved. It is crucial to proving some key results of our paper.[15]

Assumption 4 (*Technology*)**.** $1 + qc < R \leq 1/(1 - q)$.

The first inequality in the *Technology* assumption states that the net expected return on the risky asset is positive. As described, R stands for the $t = 2$ return on the risky asset, which requires one unit investment in terms of consumption goods at $t = 0$. The expected cost of restructuring is equal to qc, where c is the restructuring cost that arrives with a probability q. The second inequality, $R < 1/(1 - q)$, means that the return in the good state alone is not high enough to make banks' expected profit positive. It ensures that there is no scrapping of investment goods in the bad state.

3.1.2 Banks' problem in the bad state

Consider the problem of bank i when bad times are realized at $t = 1$. The bank has an investment level, n_i, and liquid assets of $b_i n_i$ chosen at the initial period. If $b_i \geq c$, the bank has enough liquid resources to restructure all of the assets. In this case, the bank obtains a gross return of $Rn_i + (b_i - c)n$ on its portfolio at $t = 2$. However, if $b_i < c$, then the bank does not have enough liquid resources to cover the restructuring costs entirely. In this case, the bank decides what fraction of these assets to sell $(1 - \gamma_i)$ to generate the additional resources for restructuring. Note that γ_i

[12] Gai et al. (2008) provide the leading example where this assumption is not imposed and multiple equilibria in the asset market are therefore considered. The authors assume that the choice of equilibrium is determined by the ex-ante beliefs of agents. They show that under both pessimistic and optimistic beliefs, the competitive equilibrium is constrained inefficient and exhibits overinvestment.

[13] A function is said to be log-concave if the logarithm of the function is concave. Let $\phi(y) \equiv F'(y)$ denote the (inverse) demand function of outside investors. We can rewrite this assumption as $\phi(y)\phi''(y) - 2\phi'(y)^2 \leq 0$. We can show that the demand function is log-concave if and only if $\phi(y)\phi''(y) - \phi'(y)^2 \leq 0$. Clearly the *Regularity* assumption holds whenever the demand function is log-concave. However, it is weaker than log-concavity and may also hold if the demand function is log-convex (that is, if $\phi(y)\phi''(y) - \phi'(y)^2 \geq 0$).

[14] Please see Amir (1996).

[15] Many regular return functions satisfy conditions given by the *Concavity*, *Elasticity* and *Regularity* assumptions. Here are two examples that satisfy all three of the above assumptions: $F(y) = R\ln(1+y)$ and $F(y) = \sqrt{y + (1/2R)^2}$. The following example satisfies the *Concavity* assumption, but not the *Elasticity* and *Regularity* assumptions: $F(y) = y(R - 2\alpha y)$ where $2\alpha y < R$ for all $y \geq 0$.

then represents the fraction of assets that a bank keeps after fire sales.[16] Thus, the bank takes the price of the investment good (P) as given and chooses γ_i to maximize total returns from that point on:

$$\pi_i^{Bad} = \max_{0 \leq \gamma_i \leq 1} R\gamma_i n_i + P(1-\gamma_i)n_i + b_i n_i - cn_i, \tag{1}$$

subject to the budget constraint

$$P(1-\gamma_i)n_i + b_i n_i - cn_i \geq 0. \tag{2}$$

The first term in (1) is the total return to be obtained from the unsold part of the assets. The second term is the revenue raised by selling a fraction $(1-\gamma_i)$ of the assets at the given market price, P. The third term is the liquid assets hoarded at $t = 0$. The last term, cn_i, gives the total cost of restructuring. Budget constraint (2) states that the sum of the liquid assets carried from the initial period and the revenues raised by selling assets must at least cover the restructuring costs.

By the *Concavity* assumption, the equilibrium price of assets must satisfy $P \leq F'(0) \leq R$, otherwise outside investors would not purchase any assets. In equilibrium, we must also have $P \geq c$, otherwise in the bad state banks would scrap assets rather than selling them ; that is, there would not be any fire sale. However, if there is no supply, then there is an incentive for each bank to deviate and to sell some assets to outsiders. The deviating bank would receive a price close to $F'(0)$, which is greater than the cost of restructuring, c, by assumption, as in Lorenzoni (2008). Having $P \geq c$ together with the *Technology* assumption implies that investment goods are never scrapped in equilibrium.

The choice variable, γ_i, affects only the first two terms in the expected return function of banks in (1), whereas the last terms are predetermined in the bad state at $t = 1$. The continuation return is, therefore, actually a weighted average of R and P, where weights are γ_i and $1-\gamma_i$, respectively. Banks want to choose the highest possible γ_i because they receive R by keeping assets on the balance sheet, whereas by selling them they get $P \leq R$. Therefore, banks sell just enough assets to cover their liquidity shortage, $cn_i - b_i n_i$. This means that the budget constraint binds, from which we can obtain $\gamma_i = 1 - (c - b_i)/P \in (0, 1)$. As a result, the fraction of investment goods sold by each bank is

$$1 - \gamma_i = \frac{c - b_i}{P} \in (0, 1). \tag{3}$$

The fraction of assets sold, $1-\gamma_i$, is decreasing in the price of the investment good, P, and in liquidity ratio, b_i, and increasing in the cost of restructuring, c. Therefore, the supply of investment goods

[16] Following Lorenzoni (2008) and Gai et al. (2008), we assume that banks have to restructure an asset before selling it. Basically, this means that banks receive the asset price P from outside investors, use a part, c, to restructure the asset, and then deliver the restructured assets to the investors. Therefore, banks sell assets only if P is greater than the restructuring cost, c. We could assume, without changing our results, that it is the responsibility of outside investors to restructure the assets that they purchase.

by each bank, i, is equal to

$$Q_i^s(P, n_i, b_i) = (1 - \gamma_i)n_i = \frac{c - b_i}{P} n_i \tag{4}$$

for $c \leq P \leq R$. This supply curve is downward-sloping and convex, which is standard in the fire sales literature (see Figure 2, left panel). A negative slope implies that if there is a decrease in the price of assets, banks have to sell more assets in order to generate the resources needed for restructuring. A bank's liquidity ratio, b_i, also negatively affects its asset supply in the bad state, as can be seen in (4), because a higher liquidity ratio allows a bank to offset a larger fraction of the shock using the bank's own resources.

We can substitute the optimal value of γ_i using (3) into (1) and write the maximized total returns of banks in the bad state at $t = 1$ as $\pi_i^{Bad} = R\gamma_i n_i = R(1 - \frac{c-b_i}{P})n_i$ for a given n_i and b_i. Note that the sum of the last three terms in (1) is zero at the optimal choice of γ_i because of the binding budget constraint.

3.1.3 Asset market equilibrium at date 1

We consider a symmetric equilibrium where $n_i = n$ and $b_i = b$ for all banks. Therefore, the aggregate risky investment level is given by n and the liquidity ratio is given by b as there is a continuum of banks with a unit mass. The equilibrium price of investment goods in the bad state, P, is determined by the market clearing condition

$$Q^d(P) - Q^s(P; n, b) = 0. \tag{5}$$

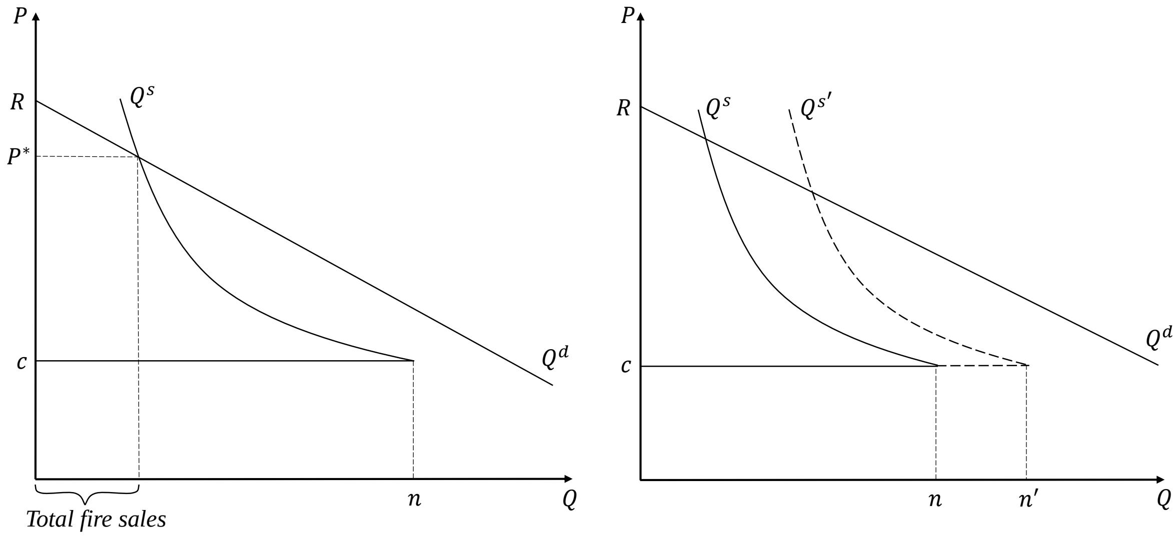
This condition says that the excess demand in the asset market is equal to zero at the equilibrium price. $Q^d(P)$ is the demand function that was obtained from the first-order conditions of the outside investors' problem, given by (3.1.1). $Q^s(P, n, b)$ is the total supply of investment goods obtained by aggregating the asset supply of each bank, given by (4).

This equilibrium is illustrated in the left panel of Figure 2. Note that the equilibrium price of the risky asset and the amount of fire sales at $t = 1$ are functions of the initial total investment in the risky asset and the aggregate liquidity ratio. Therefore, we denote the fire sale price in terms of state variables as $P(n, b)$. Lemma 1 addresses the effects of risky asset levels and the liquidity ratio on the fire sale price, while in Lemma 2, the implications for the fraction of risky asset sold is discussed.

Lemma 1. *The fire sale price of risky asset, $P(n, b)$, is decreasing in n and increasing in b.*

Lemma 1 states that higher investment in the risky asset or a lower liquidity ratio increases the severity of the financial crisis by lowering the asset prices. This effect is illustrated in the right panel of Figure 2. Suppose that the banks enter the interim period with larger holdings of risky assets.

Figure 2: Equilibrium in the investment goods market and comparative statics



In this case, banks have to sell more assets at each price, as shown by the supply function given by (4), because the total cost of restructuring, cn, is increasing in the amount of initial risky assets, n. Graphically, the aggregate supply curve shifts to the right, as shown by the dotted-line supply curve in the right panel of Figure 2, which causes a decrease in the equilibrium price of investment goods. A lower initial liquidity ratio has a similar effect by increasing the liquidity shortage in the bad state, $(c - b)n$, and hence causing a larger supply of risky assets to the market. Lower asset prices, by contrast, induce more fire sales by banks because of the downward-sloping supply curve. This result is formalized in Lemma 2.

Lemma 2. *The fraction of risky assets sold,* $1 - \gamma(n, b)$, *is increasing in* n *and decreasing in* b.

Together, Lemmas 1 and 2 imply that a higher initial investment in the risky investment by some banks, or a lower liquidity ratio, creates negative externalities for other banks by making financial crises more severe (that is, via lower asset prices, according to Lemma 1) and more costly (that is, via more fire sales, according to Lemma 2).

3.2 Competitive equilibrium

As a benchmark, we first study the competitive equilibrium. At the initial period, each bank, i, chooses the amount of investment in the risky asset, n_i, and the liquidity ratio, b_i, to maximize its expected profits:

$$\max_{n_i, b_i} \Pi_i(n_i, b_i) = (1-q)\{R + b_i\}n_i + q\{I(b_i < c)R\gamma_i + I(b_i \geq c)[R + b_i - c]\}n_i - D(n_i(1 + b_i)), \quad (6)$$

subject to the budget constraint at $t = 0$, $0 \leq (1 + b_i)n_i \leq e + E$, where $I(\cdot)$ is the indicator function and $\gamma = 1 - \frac{c-b_i}{P}$ is the fraction of assets a bank keeps after fire sales. $D(n_i(1 + b_i)) = n_i(1 + b_i) + \Phi(n_i(1 + b_i))$ is the sum of the initial cost of funds and the operational costs of a bank. Because we assume that $\Phi(\cdot)$ is convex, it follows that $D(\cdot)$ is convex as well; that is, $D'(\cdot) > 0$ and $D''(\cdot) > 0$.

Whether or not fire sales take place in the competitive equilibrium depends on the initial liquidity ratios of banks . If banks fully insure themselves against the fire sale risk—that is, if they choose $b_i \geq c$ for all $i \in [0, 1]$ at $t = 0$—then fire sales in the bad state are avoided completely. However, if banks purchase less than full insurance—that is, if $b_i < c$—then fire sales exist. The following proposition shows that in the competitive equilibrium, banks optimally choose less than full insurance and, hence, fire sales take place.

Proposition 1. *Banks take fire sale risk in equilibrium; that is, $b_i < c$ for all banks.*

Even though both the amount (c) and frequency (q) of the aggregate liquidity shock are exogenous in the model, whether and to what extent a fire sale takes place are endogenously determined. In Proposition 1 we show that perfect insurance is never optimal and that banks take some amount of fire sale risk; that is, they choose $b_i < c$. The intuition of the proof is as follows: The expected marginal return on liquid assets exceeds unity as long as there are fire sales, and it decreases with the amount of liquidity. Perfect insurance guarantees that no fire sale takes place and, as a result, the expected marginal return on liquid assets is equal to one, which is dominated by the expected marginal return on risky assets. In other words, there is no need to hoard any liquidity when there is no fire sale risk. Therefore, there is an optimal interior level of liquidity ratio for which the private marginal return and cost of liquidity are equalized.

Although banks take some fire sale risk, the main issue is whether banks take the socially optimal amount of fire sale risk. We later show that because banks do not internalize the pecuniary externalities, they end up taking too much risk. This is why bank regulation is needed in this economy. Proposition 1 allows us to focus on the imperfect insurance case; that is, $b_i < c$. First-order conditions of banks' problem under this result can be obtained as:

$$(1 - q)(R + b_i) + qR\gamma_i = D^{'}(n_i(1 + b_i))(1 + b_i), \tag{7}$$

$$(1 - q)n_i + qR\frac{1}{P}n_i = D^{'}(n_i(1 + b_i))n_i, \tag{8}$$

where $\gamma_i = 1 - \frac{c-b_i}{P}$. We denote the symmetric unregulated competitive equilibrium allocations that solve the first-order conditions (7) and (8) by n, b, and the associated price of assets in the bad state by P.

We start by solving for the competitive equilibrium price, using the first-order conditions above. Furthermore, we show in the next proposition that the closed-form solution for P is independent of the functional form of the outside investors' demand and the operational cost of banks. However, in order to solve for equilibrium investment levels and liquidity ratios, we need to make some functional-form assumptions.

Proposition 2. *The competitive equilibrium price of assets is given by*

$$P = \frac{qR(1+c)}{R-1+q}. \tag{9}$$

The equilibrium price, P, is increasing in the probability of the liquidity shock, q, and the size of the shock, c, but decreasing in the return on the risky assets, R.

Proposition 2 shows that the price of assets in the bad state increases in the expected size of the liquidity shock, qc. When banks expect to incur a larger additional cost for the investment, or when they face this cost with a higher probability, they reduce risky investment levels and increase liquidity buffers, as we show in the next proposition. As a result, there are fewer fire sales and a higher price for risky assets in the competitive equilibrium. Proposition 2 implies that banks act less prudently (by increasing risky investment and reducing liquidity) if they expect financial shocks to be less frequent (a lower q), which in turn leads to more severe disruption to financial markets (through lower asset prices and more fire sales) if shocks do materialize. Stein (2012) obtains a similar result as well.[17]

3.2.1 A closed-form solution for the competitive equilibrium

In order to obtain closed-form solutions for the equilibrium values of n and b, we need to make functional-form assumptions for outside investors' production technology, F, and the operational cost of banks, Φ. Suppose that the operational costs of a bank are given by $\Phi(x) = dx^2$, and hence $\Phi'(\cdot)$ is increasing; that is, $\Phi'(x) = 2dx$. Note that marginal cost of funds is increasing in parameter d. On the demand side, suppose that the outside investors' return function is given by $F(y) = R\ln(1+y)$. It is easy to verify that this function satisfies the *Concavity*, *Elasticity*, and *Regularity* assumptions. In Section 7.1 in the Appendix, we solve for the competitive equilibrium investment level and liquidity ratios, as follows:

$$n = \frac{\tau}{\tau+1}\frac{q(\tau+1)+2dR}{2d(1+c)}, \qquad b = \frac{cq(\tau+1)-2dR}{q(\tau+1)+2dR},$$

where

$$\tau \equiv \frac{R}{P} - 1 = \frac{R-1+q}{q(1+c)} - 1.$$

Proposition 3 presents the comparative statics of the liquidity ratio and risky investment level with respect to model parameters in the competitive equilibrium.

[17] This result is reminiscent of the financial instability hypothesis of Minsky (1992), who suggests that "over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system."

Proposition 3. *The comparative statics for the competitive equilibrium risky investment level, n, and liquidity ratio, b, are as follows:*

1. *The risky investment level (n) is increasing in the return on the risky asset (R) and decreasing in the size of the liquidity shock (c), probability of the bad state (q), and the marginal cost parameter (d).*

2. *The liquidity ratio (b) is increasing in the return on the risky asset (R), size of the liquidity shock (c), and the probability of the bad state (q), and decreasing in the marginal cost parameter (d).*

Proposition 3 shows that b and n move in the same direction in response to R and d, while they move in opposite directions in response to c and q. Risky and liquid assets can increase or decrease simultaneously as a response to a change in R and d, thanks to the flexible bank balance sheet size. This result is intuitive because cq is the expected value of the liquidity need at the interim period, and as this need increases, the bank holds more liquidity and fewer risky assets. Of course, this result does not say whether the bank increases its liquidity ratio sufficiently from a socially optimal perspective.

3.3 Constrained planner's problem

In this section, we consider the problem of a constrained planner who is subject to the same market constraints as the private agents. In particular, the planner takes the limited commitment in financial contracts between banks and outside investors as given. However, unlike banks, the constrained planner takes into account the effect of initial portfolio allocations on the price of assets in the bad state. The constrained planner chooses both the risky investment level, n, and the liquidity ratio, b, at $t = 0$ to maximize the net expected social welfare which consists of expected bank profits, the expected utility of a representative consumer and the expected profits of outside investors:

$$\max_{n,b} W(n,b) = (1-q)\{R+b\}n + q\{I(b<c)R\gamma + I(b \geq c)[R+b-c]\}n - D(n(1+b)) \\ +qI(b<c)[F((1-\gamma)n) - P(1-\gamma)n] + e, \qquad (10)$$

subject to $0 \leq (1+b)n \leq e+E$, society's budget constraint at $t = 0$, where $I(\cdot)$ is the indicator function and $\gamma = 1 - \frac{c-b}{P}$. Note that depositors are risk-neutral and consume $e - L$ units at $t = 0$ and L units at $t = 2$ in both states because deposits are safe and produce a unit return. Hence, their expected utility at $t = 0$ is equal to $(e - L) + L = e$. The term, $F((1-\gamma)n) - P(1-\gamma)n$, gives the profits of outside investors in case of fire sales: Outside investors purchase $(1-\gamma)n$ units of risky assets from banks at the market price P, and produce a yield of $F((1-\gamma)n)$ from these assets in the final period. The first question is whether the constrained planner would avoid fire sales completely by setting $b \geq c$. The next proposition addresses this question.

Proposition 4. *It is optimal for the constrained planner to take fire sale risk; that is, the constrained optimal liquidity ratio satisfies $b < c$.*

The proposition states that it is optimal for the constrained planner to expose the banking sector to some amount of fire sale risk. In other words, full insurance is not constrained optimal. A higher liquidity ratio decreases fire sales by decreasing the liquidity shortage of each bank (microprudential) and by increasing the price of the risky asset (macroprudential). That is, holding liquidity makes each bank less exposed to the fire sale risk, and the fire sales are less severe as a result of higher fire sale prices. However, the marginal benefit decreases with the amount of liquidity at both the bank and aggregate levels. Meanwhile, the opportunity cost of holding liquidity is the foregone profit from not investing in risky asset. The constrained social planner weighs the opportunity cost against the microprudential and macroprudential benefits of liquidity in the bad state and determines the optimal amount of fire sale risk to take.

In the proof of Proposition 4 we show that full insurance is not optimal as long as the price of assets does not decrease severely as a result of a small amount of asset sales. A smooth demand curve with an intercept close to R is a sufficient condition to get this result. For example, setting $F'(0) = R$ would be sufficient, however, the necessary condition is much looser, as shown in the proof.

Proposition 4 allows us to focus on the $b < c$ case when analyzing the constrained planner's problem given by (10). Corresponding first-order conditions with respect to n and b are, respectively,

$$(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial \gamma}{\partial n}n\Big\} + q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right) - c + b\right\} = D^{'}(n(1+b))(1+b), \quad (11)$$

$$(1-q)n + qR\Big\{\frac{1}{P} + \frac{c-b}{P^2}\frac{\partial P}{\partial b}\Big\}n + q\left\{F'((1-\gamma)n)\left(-\frac{\partial \gamma}{\partial b}\right)n + n\right\} = D^{'}(n(1+b))n, \quad (12)$$

where $\gamma = 1 - \frac{c-b}{P}$. We denote the constrained efficient allocations that solve the first-order conditions (11) and (12) by n^{**}, b^{**}, and the associated price of assets in the bad state by P^{**}. Section 7.2 in the Appendix presents the closed-form solutions for n^{**}, b^{**} and P^{**}.

These first-order conditions differ from the first-order conditions of the banks' problem in Section 3.2 in two aspects. First, there are extra terms because the social welfare function includes not only the total profit of banks but also the expected utility of depositors and the expected profits of outside investors. Second, and more importantly, unlike the individual banks, the constrained planner takes into account how changing the initial risky investment level and liquidity ratio affects the price of assets, P, and the fraction of assets sold to outside investors, $1-\gamma$. In other words, the constrained social planner internalizes the fire sale externalities, that is, the planner internalizes the fact that larger risky investments or lower liquidity ratios lead to a lower asset price and more fire sales in the bad state.

Comparing the first order conditions from the competitive equilibrium to the ones from the planner's problem does not seem straightforward yet the comparison can easily be simplified at no cost. We can drop profits of outside investors and expected utility of depositors from the social welfare function and allow the pecuniary externality to be the only difference between the banks' problem in the unregulated competitive equilibrium and the constrained planner's problem. This setup allows us to isolate the effect of the fire sale externality on the equilibrium outcomes. In that case, when we compare the first-order condition with respect to n between the two cases, given by (7) and (11), the only extra term in the constrained planner's problem would be $qR\frac{\partial \gamma}{\partial n}n$. This term is negative and captures the extra units of fire sales by other banks, caused by each banks' additional investment in the risky asset. Similarly, when comparing (8) and (12), the only extra term in constrained planner's problem would be $qR\frac{c-b}{P^2}\frac{\partial P}{\partial b}n$, which is positive. This term captures the public good property of liquidity: The liquid asset held by banks not only insures them against the fire sale risk but also constitutes a positive externality on other banks via greater fire sale prices.

In the online appendix we show that all results of the paper are the same when the sole difference between the two problems is the pecuniary externality. Thus, we can assert that our results are driven by the pecuniary externality, not by the differences in the objective functions between the unregulated case and constrained planner's problem. Note that the assumption that the planner does not incorporate the profit of outside investors and depositors can be justified when the regulator is solely concerned or responsible with the well-being of bankers. Alternatively, aware of the fire sale externality in the decentralized setup, banks can create a self-regulatory institution that will improve their well-being by internalizing the externality.

The reason why incorporating the outside investors into the social welfare function does not make a difference is as follows. Fire sales are costly because assets are reallocated from more efficient bankers to less efficient outside investors through fire sales. Both bankers and outside investors make profits by managing the risky asset, and the profits of outside investors is lower. Incorporating the profits of outsider investors decreases the social cost of fire sales, and changes the initial portfolio choices, however, it does not eliminate the social cost of fire sales. Therefore, we obtain the same results whether we incorporate the profits of social investors into the social welfare function or not.

In the next proposition, we compare the competitive equilibrium level of risky assets and liquidity ratios with the constrained efficient allocations. To perform the comparison, we use the closed-form solutions of equilibrium outcomes presented in the appendix.

Proposition 5. *Competitive equilibrium allocations compare to the constrained efficient allocations as follows:*

1. *Risky investment levels:* $n > n^{**}$
2. *Liquidity ratios:* $b < b^{**}$

Proposition 5 shows that in the competitive equilibrium, unregulated banks overinvest in the risky asset, $n > n^{**}$, and inefficiently insure against liquidity shocks by holding low liquidity ratios, $b < b^{**}$. The inefficiency of the competitive equilibrium allocations is due to a combination of banks' failure to internalize the effects of initial portfolio choices on prices and limited-commitment problem that prevents banks from raising external finance in the bad state. As a result, the contraction in risky investment and decrease in asset price are excessive when the liquidity shock is realized and there is no regulation on banks.

The first result is reminiscent of Lorenzoni (2008) and Korinek (2011), who show that there is excessive risky investment under fire sale externalities. The latter, meanwhile, is reminiscent of Bhattacharya and Gale (1987) and Allen and Gale (2004b), who show that private holdings of liquid assets are inefficient under incomplete markets. Allowing banks to invest in both the risky illiquid asset and liquid asset, we show that the pecuniary externality manifests itself in both choices of the banks and distorts both margins. Together with the flexible balance sheet size, this setup allows us to study the interaction between the two as well.

3.4 Implementing the constrained efficient allocations: complete regulation

In Proposition 5, we have shown that the socially optimal risky investment level is lower than the privately optimal level and the socially optimal liquidity ratio is higher than the privately optimal ratio because of the existence of pecuniary externalities. Therefore, the constrained efficient allocations can be implemented by applying simple quantity regulations to banks. In particular, a regulator can implement the optimal allocations (n^{**}, b^{**}) by imposing a minimum liquidity ratio as a fraction of risky assets ($b_i \geq b^{**}$) and a maximum level of risky investment ($n_i \leq n^{**}$). The latter corresponds to a minimum risk-weighted capital ratio; that is, $E/n_i \geq E/n^{**}$.

These quantity-based rules can be mapped to the capital and liquidity regulations in the Basel III accord. First, the risk-weighted capital ratio, E/n_i, corresponds to the Basel definition, as it gives liquid assets, $n_i b_i$, a zero risk weight while giving risky assets, n_i, a weight of one in the denominator. In reality, banks carry several risky assets on their balance sheet for which Basel Accords require different risk weights. However, introducing assets with different risk profiles to our setup would complicate the analysis without adding further insight. Second, our liquidity regulation mimics the liquidity coverage ratio requirement proposed in Basel III. The LCR requires banks to hold high-quality liquid assets against the outflows expected in the next 30 days. In our setup, the expected cash outflow is the expected liquidity need, qc, per each risky asset. Therefore, the liquidity requirement in our setup can be equivalently written as $b_i n_i / qcn_i \geq b^{**} n^{**} / qcn^{**}$. It is true that the LCR focuses on liquidity shocks on the liability side whereas here we consider liquidity shocks on the asset side. However, this modeling choice is not essential to our result; all we need is a liquidity requirement in some states of the world that cannot be fully met with raising external finance. If we instead model liquidity shock as a proportion of deposits, we would then need capital regulation to limit the size of deposits and liquidity requirement to increase the high quality liquid assets (cash). Although we have a stylized model, the time frame between the two model periods can be calibrated according to Basel definitions as well.

The liquidity requirement was missing in the pre-Basel III era. In order to understand whether Basel III regulations are a step in the right direction, one needs to compare them to the pre-Basel III era. For this purpose, in the next section we study a regulated economy that is similar to the Basel I and Basel II eras; that is, the capital ratios of banks are regulated but there is no requirement on the banks' liquidity. Hence, we consider banks that are free to choose their liquidity ratios for a given capital requirement. This setup also allows us to study the interaction of banks' capital and liquidity ratios and to provide an answer to Tirole's question quoted in the introduction: What happens to banks' liquidity when their capital ratios are regulated? Do banks manage their liquidity in an efficient way, or does capital regulation distort their choice of liquidity?

4 Partial regulation: regulating only capital ratios

In this section, we consider the problem of a regulator who chooses the level of risky investment, n, at $t = 0$ to maximize the net expected social welfare but who allows banks to freely choose their liquidity ratio, b_i. In the next section, we show that the optimal risky investment level in this case is lower than the competitive equilibrium level. As a result, the regulator can implement the optimal level by introducing it as a regulatory upper limit on domestic banks' risky investment level, which corresponds to the risk-weighted minimum capital ratio requirement in our fixed amount of bank equity framework. We consider this case to mimic the regulatory framework in the pre-Basel III period, which predominantly focused on capital adequacy requirements. We first analyze the problem of a representative bank for a given regulatory investment level, n. The bank chooses the liquidity ratio, b_i, to maximize its expected profits; hence, the problem of the bank is as follows:

$$\max_{b_i} \Pi_i(b_i; n) = (1-q)\{R + b_i\}n + qR\gamma_i n - D(n(1+b_i)). \tag{13}$$

The first-order condition of the banks' problem (13) with respect to b_i is

$$1 - q + qR\frac{1}{P} = D^{'}(n(1+b_i)).$$

From this first-order condition, we can obtain banks' (implicit) reaction function to the regulatory investment level—that is, the liquidity ratio, b_i, that banks choose for each given risky investment level, n—as follows:

$$b_i(n) = \frac{D^{'-1}(1 - q + q\frac{R}{P})}{n} - 1. \tag{14}$$

The regulator takes this reaction function into account while choosing the optimal risky investment level to maximize the expected social welfare:

$$\max_{n} \; W(n) = \; (1-q)\{R+b(n)\}n + qR\gamma n - D((1+b(n))n) + q[F((1-\gamma)n) - P(1-\gamma)n] + e.$$

The social welfare is equal to the sum of expected bank profits, the expected utility of a representative consumer and the expected profits of outside investors. Unlike an unconstrained social planner, the regulator is subject to the same constraints as the private agents. In particular, the regulator takes the limited commitment of banks in financial contracts with outside investors as given. Similar to the discussion in Section 3.3, the main difference of this regulatory objective function from the banks' problem given by (6) is that the regulator takes into account the effect of the initial risky investment level on the price of assets in the bad state. The optimal risky investment level in this case is determined by the following first-order condition of the regulator's problem with respect to n:

$$(1-q)\{R+b(n)+nb^{'}(n)\} + qR\left\{\gamma + n\frac{d\gamma}{dn}\right\} + q[F'(\cdot)\left(1-\gamma-\frac{d\gamma}{dn}n\right) - c + b(n) + nb'(n)] = D^{'}(n(1+b))\{1+b(n)+nb^{'}(n)\}. \quad (15)$$

We denote the optimal risky investment level that solves the first-order condition (15) by n^*, the associated optimal liquidity choice of banks under partial regulation by b^*, and the price of assets in the bad state by P^*. Section 7.3 in the Appendix presents the closed-form solutions for n^*, b^*, and P^*.

Changing n has an indirect effect on the asset price in the bad state in addition to its direct effect because banks change their liquidity ratios in response to changes in n, which then changes the price of assets in the equilibrium. The main question in this case is how banks respond to a tightening of capital regulations. We answer this question in Proposition 6.

Proposition 6. *Let the operational cost of a bank be given by $\Phi(x) = dx^2$. Then, for any technology function for outside investors, $F(\cdot)$, that satisfies the Concavity, Elasticity, and Regularity assumptions with $F'(0) = R$, banks decrease their liquidity ratio as the regulator tightens capital requirements; that is, $b^{'}(n) \geq 0$.*

Proposition 6 shows that banks reduce their liquidity ratios as the regulator tightens the risky investment level. The regulator attempts to correct banks' excessive risk-taking by introducing a risk-weighted capital ratio requirement. However, because this regulation prevents banks from reaching their privately optimal level of risk, they react by reducing their liquidity ratios. In other words, banks undermine the purpose of capital regulations by carrying less-liquid portfolios. It would not be surprising to observe banks holding fewer liquid assets when they are asked by the regulator to decrease their risky asset holdings. However, what is stated in Proposition 6 goes beyond that: Banks also decrease their liquidity ratios when the the regulator limits the risky investment level; that is, banks hoard less liquidity per unit of risky asset.

The intuition of the proof is as follows: The marginal return to the liquidity ratio, b_i, is $(1-q)n_i + q\frac{R}{P}n_i$, and it is decreasing in the fire sale price, P: Each unit of liquidity holding per risky asset becomes less valuable as the fire sale price increases. Capital regulation lowers the amount of risky investment, and hence increases the fire sale price as we show in Lemma 1. As a result, liquidity hoarding becomes less attractive for the banks, and they decrease their liquidity ratio, b_i. In other words, the benefit of holding liquidity is to be less exposed to fire sale risk. A higher fire sale price (induced by capital regulations) means that the risk is less costly, and thus it is optimal for banks to take more of that risk (decrease b_i). The fact that banks can decrease their liquid assets at the same time when capital regulation is limiting their risky assets is possible due to the flexible balance sheet size of banks in our model.

Note the role of rational expectations in generating this relationship between capital regulation and liquidity ratios. Because the regulation applies to every bank, banks correctly forecast the fire sale price to be higher as a result of less risky investment in the banking system, and they optimally decrease their liquidity ratios. In a sense, Proposition 6 reveals an unintended consequence of capital regulation when it is applied in isolation. If the financial system is now more stable—that is, if there are higher fire sale prices and fewer fire sales—banks' incentive to hoard liquidity is smaller.

We can also use an analogy from automobile safety regulations to explain the result in Proposition 6. Peltzman (1975) and Crandall and Graham (1984) show that whether automotive safety regulations such as safety belts and airbags reduce the fatality rate depends upon the response of drivers to the increased protection. The empirical evidence they present shows that drivers do indeed increase their driving intensity as a response to safety regulations, resulting in a less than expected reduction in fatality rates. Similarly, in our setup, capital regulations intend to make the financial system safer, but individual banks respond by taking more risk on the liquidity channel. As a result, there is a less than expected increase in financial stability and welfare from regulation.

Proposition 6 states that for any elastic, log-concave demand function with the intercept $F'(0) = R$, the result $b'(n) \geq 0$ holds. However, the condition is actually looser than that because, as we have discussed, the *Regularity* assumption is weaker than log-concavity. In the appendix, we show that it is even possible to relax this constraint further and obtain this result with weak inequality—that is, $F'(0) \leq R$—if we make the following assumption on outside investors' technology:

$$R < \frac{F'(F' + yF'')}{F' + 2yF''}.$$

The proof provides a sufficient condition by showing that there is strategic complementarity between the regulatory risky investment level, n, and the liquidity ratio, b_i, for each bank. We use the monotone comparative statics techniques outlined by Vives (2001) to show that banks' profit function exhibits increasing differences between n and b_i in the partial regulation case.

4.1 Complete versus partial regulation: do we need liquidity requirements?

In this section, we investigate whether capital regulation alone can restore constrained efficiency. For this reason, we compare the equilibrium outcomes (level of risky assets, liquidity ratios, asset prices, and the amount of fire sales) in three different settings: a decentralized equilibrium without any regulation, a partially regulated case in which there is only capital regulation, and a complete regulation case in which the constrained optimum is achieved using both capital and liquidity regulations. To perform the comparison, we use the closed-form solutions of equilibrium outcomes presented in the appendix. Proposition 7 summarizes the results.

Proposition 7. *Risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium* $(n, b, P, 1-\gamma, (1-\gamma)n)$*, partial regulation equilibrium* $(n^*, b^*, P^*, 1-\gamma^*, (1-\gamma^*)n^*)$*, and complete regulation equilibrium* $(n^{**}, b^{**}, P^{**}, 1-\gamma^{**}, (1-\gamma^{**})n^{**})$ *compare as follows:*

1. *Risky investment levels:* $n > n^{**} > n^*$
2. *Liquidity ratios:* $b^{**} > b > b^*$
3. *Financial stability measures*
   (a) *Price of assets in the bad state:* $P^{**} > P^* > P$
   (b) *Fraction of assets sold:* $1-\gamma > 1-\gamma^* > 1-\gamma^{**}$
   (c) *Total fire sales:* $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

In a partially regulated financial system, unlike the competitive economy, the overinvestment problem does not arise. On the contrary, in Proposition 7 we show that the investment in risky assets under partial regulation is inefficiently low compared to the constrained optimum: $n^* < n^{**}$. The underinvestment is related to the liquidity choice of banks: The problem of unregulated banks having inefficiently low liquidity ratios is exacerbated with the introduction of capital regulation in isolation. In Proposition 7 we show that banks are less liquid under partial regulation than they were in the competitive equilibrium, that is, $b^* < b$. As discussed in Proposition 6, as capital regulation limits the risky investment, banks optimally choose less-liquid portfolios, which partially offsets the positive impact of the reduction in risky investment on financial stability and welfare. Lower liquidity ratios expose the financial system to excessive fire sales and asset price decreases. The precautionary behavior of the regulator is then to implement the capital regulation in a more restrictive way, which increases the fire sale price but leads to an inefficiently low level of investment in the profitable risky asset.

Higher liquidity ratios are why more investment in long-term assets is allowed under complete regulation. Proposition 7 shows that the constrained optimal level of liquidity, b^{**}, is higher than the liquidity chosen by banks under the partial regulation, b^*. Higher liquidity ratios allow banks to hold more risky assets without increasing the fire sale risk. Therefore, the socially optimal choice is to hold a higher level of risky investment supported by greater liquidity ratios.

In order to see the interaction between the capital and liquidity requirements, consider the following scenario: A country transitions from partial regulation to complete regulation by imposing new liquidity rules in addition to existing capital rules. To be specific, this transition can be compared to moving from the Basel I/II regulatory approach to the Basel III regulatory approach. *Assuming that capital regulation had been set optimally during the pre-Basel III period*, capital requirements can be relaxed after the introduction of liquidity requirements. Therefore, our results would predict that more long-term profitable risky investment can be financed via the banking system after the implementation of liquidity requirements.

How effective is capital regulation in addressing financial instability caused by fire sales when it is not accompanied by liquidity requirements? To answer this question, we can compare the measures of financial instability across the two regulatory regimes. More fire sales and a lower price of the risky asset in the bad state are associated with greater financial instability, and they imply that the externality has a stronger presence in the economy. Proposition 7 shows that the introduction of capital regulation in isolation increases the fire sale price compared to the competitive equilibrium price. However, the price is still below the socially optimal price level, which can only be achieved with the addition of liquidity requirements. The message is the same when we compare both the fraction and the total amount of risky assets that must be sold to withstand the liquidity shock under the two regulatory regimes, as shown in items 3-a and 3-b in Proposition 7. In general, minimum capital rules may actually serve several purposes, such as countering moral hazard problems generated by the existence of limited liability and deposit insurance, that we do not analyze in this paper. However, what we show here is that, under fire sale externalities, capital regulations are not effective in achieving constrained optimum unless they are combined with liquidity requirements.

Our results indicate that neither capital nor liquidity ratios alone are perfect predictors of potential instability; a better-capitalized banking system may end up conducting larger fire sales. Under partial regulation, for instance, although the capital ratios are higher than under complete regulation, the liquidity shock causes a larger disruption to financial markets. Similarly, a more-liquid banking system may experience greater financial instability; banks are more liquid in the unregulated competitive equilibrium compared with partial regulation, but the shock leads to more distortions in the former.

We end this section by comparing bank size across three different regimes in the following proposition, and we discuss the implications of this result for simple leverage ratio regulation.

Proposition 8. *Bank balance sheet sizes across different regimes are as follows:*

$$n(1+b) = n^{**}(1+b^{**}) > n^{*}(1+b^{*})$$

Proposition 8 shows that the bank size in the competitive equilibrium is equal to the socially optimal size. However, bank size is inefficiently small under partial regulation as there are both lower risky and liquid assets in this regime compared to the constrained optimum. Proposition 8 provides an interesting result on the regulation of leverage ratio, which can be defined as $\frac{E}{n(1+b)}$ in this setup. Proposition 8 shows that the optimal simple leverage ratio is the same under complete regulation and unregulated competitive equilibrium. In our setup, this proposition means that starting from the competitive equilibrium, it is possible to obtain the *constrained social optimum* by reallocating funds from risky assets to liquid assets without affecting the balance sheet size. However, as shown in Proposition 7, the two cases are different in terms of financial stability implications. This suggests that, from a financial stability perspective, what is important is not the size of a bank's balance sheet but its composition. Therefore, we can conclude that in the current setup, a leverage regulation applied in isolation would be completely ineffective.[18] However, the leverage regulation combined either with a liquidity ratio requirement or with a risk-weighted capital regulation would be sufficient to replicate the *constrained social optimum*.

4.2 Can regulating only liquidity be the solution?

In our model, fire sales are triggered by a liquidity shock in the bad state. Banks are solvent as long as they can cover this liquidity requirement because the return on the risky asset (R) is greater than the cost of restructuring needed to keep the investment alive (c) by the *Technology* assumption. Therefore, one may wonder if the constrained optimum can be implemented using liquidity regulation alone, that is, without using capital requirements at all. The short answer is no. First, note that, in Proposition 4, we show that it is not optimal to avoid fire sales completely in the bad state by forcing banks to perfectly insure against the liquidity shock by setting $b = c$. Second, regulating only liquidity means that banks are free to choose their capital ratios. The questions then becomes whether banks choose the optimal capital ratio when the minimum liquidity requirement is set optimally.

In order to answer this question, we first need to study bank behavior under liquidity regulation alone. In this case, the regulator chooses the optimal liquidity ratio, b, at $t = 0$ to maximize the net expected social welfare but allows banks to freely choose their risky investment level, n_i. Consider the problem of a bank first: For a given regulatory liquidity ratio, b, a bank chooses the level of risky investment, n_i, to maximize its expected profits:

$$\max_{n_i} \Pi_i(n_i; b) = (1-q)(R+b)n_i + qR\gamma n_i - D(n_i(1+b)). \tag{16}$$

The first-order condition of the banks' problem (16) with respect to n_i is

$$(1-q)(R+b) + qR\gamma - D^{'}(n_i(1+b))(1+b) = 0. \tag{17}$$

[18] Nevertheless, leverage ratio regulation might be an important method of addressing other market failures, such as risk shifting or informational asymmetries, which we do not study in this model.

From this first-order condition, we can obtain banks' reaction function, $n_i(b)$, to the regulatory liquidity ratio—that is, the optimal risky investment level, n_i, that banks choose for each given liquidity ratio, b. Now, using banks' optimal response function, $n_i(b)$, we can check if banks choose the constrained optimal risky investment level, n^{**}, if the regulator sets the minimum liquidity ratio at the constrained optimal level, b^{**}. That is, can we have $n_i(b^{**}) = n^{**}$? The next lemma answers this question:

Proposition 9. *Banks do not choose the constrained optimal risky investment level, n^{**}, if the regulator sets the minimum liquidity ratio at the constrained optimal level, b^{**}, that is, $n_i(b^{**}) \neq n^{**}$.*

Proposition 9 states that banks do not choose the optimal capital ratio when they are asked by the regulator to hold the optimal liquidity ratio. In fact, in the appendix we show that they choose a higher level than the optimal level of risky investment, that is, $n_i(b^{**}) > n^{**}$. Therefore, the constrained planner's allocations cannot be implemented by regulating liquidity alone. Banks can take on the fire sale risk through both liquidity and capital channels. Therefore, implementing constrained efficiency requires restraining banks on both channels. Otherwise, banks use the unregulated channel to take more risk, undermining the regulator's intent to correct for the fire sale externality.

4.3 Further Policy Implications

Central banks and regulatory institutions around the world mainly focus on regulating banks to improve financial stability. However, actions of nonbank financial institutions affect the stability of the system as well. Yet, some financial institutions are partially or totally exempt from bank regulations. For instance, in the U.S. hedge funds and investment banks do not need to comply with Basel regulations, while small banks (banks with less than $50 billion in total assets) are exempt from the Basel III liquidity requirement LCR.[19] Nevertheless, institutions that are outside the scope of bank regulation and supervision are indirectly affected by the regulatory rules. One such indirect channel arises because unregulated institutions invest in similar risky assets and trade in the same common asset markets as regulated banks. For instance, hedge funds or investment banks hold mortgage-backed securities in their portfolios.

The portfolio allocations of unregulated institutions matter for the fire sale market because the sale of these risky assets in distress times contribute to the deterioration of fire sale prices. Moreover, in case of fire sale externalities, regulations that require banks to be more prudent could, at the same time, create incentives for unregulated institutions to take on more fire sale risk. Therefore, from the perspective of bank regulation, it is important to understand the reaction of unregulated institutions to bank regulation. For example, ignoring the reaction of unregulated institutions could

[19] http://www.federalreserve.gov/newsevents/press/bcreg/20140903a.htm.

lead to over or under-regulation of banks. Such spillover effects from unregulated institutions could hamper the ability of regulators to achieve greater financial stability and welfare.

We analyze how unregulated financial institutions react to bank regulation as well as what their reactions imply for financial stability. For that purpose, we introduce unregulated financial institutions into our model. These institutions are identical to banks other than being exempt from the regulatory requirements. As mentioned above, in case of liquidity regulation these institutions can be considered as small banks. This simple setup allows us to study the implications of the interaction between regulated and unregulated institutions with minimal modification of our model.

We denote the choices of regulated institutions with $(\tilde{n}, \tilde{b})$ and those of unregulated institutions with (n, b). As before, n and $\tilde{n}$ are the amount of risky investment while b and $\tilde{b}$ denote the liquid asset per unit risky asset. Liquidity needs of regulated and unregulated institutions in bad times at $t = 1$ respectively are: $(c - \tilde{b})\tilde{n}$ and $(c - b)n$. The market clearing condition in the fire sale market is $(c - \tilde{b})\tilde{n} + (c - b)n = Py$. Thus, the fire sale price is a function of $\tilde{n}, \tilde{b}, n, b$. Below we analyze the response of unregulated institutions to bank regulation. In particular, we study the risky asset choice of an unregulated institution and see how it changes as the regulator limits the total risky investment $\tilde{n}$, in the regulated segment. An unregulated institution chooses n_i, b_i to maximize its expected profits, given by:

$$\Pi(n_i, b_i) = \; (1 - q)\{R + b_i\}n_i + qR\gamma_i n_i - D((1 + b_i)n_i)$$

where $\gamma_i = 1 - (c - b_i)/P$. Here, the atomistic institution takes the fire sale price $P(\tilde{n}, \tilde{b}, n, b)$ as given and we treat $\tilde{n}$ as a parameter of the model because unregulated institutions take it as given. The regulator effectively determines the aggregate amount of $\tilde{n}$ using capital regulations. Therefore, the first-order condition of the unregulated institution with respect to n_i is

$$\frac{\partial \Pi(n_i, b_i)}{\partial n_i} \;=\; (1 - q)(R + b_i) + qR\gamma_i - D'(\cdot)(1 + b_i) = 0$$

The first order-condition above determines the level of optimal risky investment for an unregulated institution, n_i, for a given level of aggregate risky investment in the regulated segment, $\tilde{n}$. In order to see how n_i changes with $\tilde{n}$ we need to evaluate the sign of the cross-partial derivative of the profit function:

$$\frac{\partial^2 \Pi(n_i, b_i)}{\partial \tilde{n} \partial n_i} = qR\frac{\partial \gamma}{\partial P}\frac{\partial P}{\partial \tilde{n}} < 0.$$

The higher the level of risky investment allowed by the regulator, the lower the fire sale price is, and that is captured by negative $\frac{\partial P}{\partial \tilde{n}}$. The lower fire sale price, in turn, leads to lower level of risky asset left at the bank after the fire sales, which is captured by positive $\frac{\partial \gamma}{\partial P}$. Altogether, using the monotone comparative statics techniques outlined by Vives (2001), the negative sign of the cross-partial derivative indicates that $n'(\tilde{n}) < 0$, that is, as regulation tightens risky investment level of banks, unregulated institutions respond by increasing their risky investment. Using the terminology of monotone comparative statics, n_i and $\tilde{n}$ are strategic, yet imperfect, substitutes from the unregulated institution's point of view.

Similarly, we evaluate how unregulated institutions respond to tighter liquidity regulations.

$$\frac{\partial \Pi(n_i, b_i)}{\partial b_i} = (1-q)n_i + qRn_i\frac{1}{P} - D'(\cdot)n_i$$

The first order-condition above determines the optimal liquidity ratio for an unregulated institution, b_i, for a given level of liquidity ratio in the regulated segment, $\tilde{b}$. In order to see how b_i changes with $\tilde{b}$ we evaluate the sign of the cross-partial derivative of the profit function:

$$\frac{\partial^2 \Pi(n_i, b_i)}{\partial \tilde{b} \partial b_i} = -qRn_i\frac{1}{P^2}\frac{\partial P}{\partial \tilde{b}} < 0.$$

The sign of this derivative is negative by Lemma 1. The negative sign on the cross-partial derivative shows that $b'(\tilde{b}) < 0$, that is, as the regulation require (some) banks to increase their liquidity ratios, unregulated institutions respond by decreasing their liquidity ratios. Thus, unregulated institutions free ride on the liquidity of regulated institutions.

In similar ways, we can also show that unregulated institutions increase the level of their risky investment as the regulation requires more liquidity, that is $n'(\tilde{b}) < 0$, and they decrease their liquidity buffers with the regulation on the amount of risky investment, that is, $b'(\tilde{n}) < 0$. Regulations on $\tilde{n}$ and $\tilde{b}$ make the financial system more stable by increasing the fire sale price, which in turn create incentives for the unregulated institutions to invest more in risky assets and decrease their liquidity buffer. The behavior of unregulated institutions creates a counter force to the regulation.

To explain the intuition behind these results, we can consider another analogy from automotive safety regulations in the spirit of Peltzman (1975): Cars and motorcycles usually share the same roads. If we introduce speed restrictions on cars but not on motorcycles, roads will initially become safer, but this will create incentives for motorcycle riders to increase their driving intensity, creating a counter force to the regulation.

The effect analyzed in this section is similar to the one examined in international policy coordination literature such as Acharya (2003), Dell'Ariccia and Marquez (2006), and Kara (2015). These papers show that bank regulations across countries are strategic substitutes. For instance, Kara (2015) shows that if the regulator of one country tightens capital regulations, the regulator of the other country finds it optimal to relax its capital regulations and allow banks in its jurisdiction to invest more in risky assets. This result is driven by the public good property of capital regulations in an international context under fire sale externalities. Similarly, we show above that even in a given country bank capital and liquidity regulations have a public good property under fire sale externalities. If we regulate only some institutions, unregulated institutions that engage in similar investment behavior will free ride on the improved stability brought by disciplined institutions. Therefore, as argued by Farhi et al. (2009), efficient regulations should have a wide scope and apply to all relevant financial institutions.

5 Discussion of assumptions

In this section, we show that our results are robust to some changes in the modeling environment. First, we consider deposit markets that have monopolistic competition and endogenize the deposit rate while assuming that banks have limited liability. In previous sections, we assume that the bank equity is sufficiently large to prevent banks from defaulting in the bad state; as a result, each bank can raise deposits from consumers at a net zero interest rate. This new setup allows default in equilibrium. Second, we introduce deposit insurance and limited liability together. We do not need the limited liability or deposit insurance assumptions in the basic setup because there is no default in equilibrium. We show that the results of the paper do not change under these different modeling environments. This is because the constrained inefficiency of the decentralized equilibrium, and hence the justification for capital regulations and liquidity regulations, does not depend on a moral hazard problem created by the existence of deposit insurance or limited liability for bank owners. Instead, the inefficiency depends purely on the existence of pecuniary externalities under incomplete markets. In other words, the inefficiency driven by fire sale externalities would prevail in a narrow banking system in which banks are all financed by nothing but equity capital.

Next, we discuss relaxing the convex operational cost assumption and its implications for our results. Last, we show that the aggregate nature of the liquidity shock is not material for the mechanism or the conclusions of the model. We allow the liquidity shock to be idiosyncratic rather than aggregate and show that this setup is isomorphic to the aggregate shock case with a smaller liquidity shock.

5.1 Endogenizing the deposit rate

In previous sections, we assumed that bank equity is sufficiently large to prevent banks from default in the bad state and, as a result, that each bank could raise deposits from consumers at a net zero interest rate. Now, instead, suppose that each bank is a local monopsony in the deposit market and there is limited liability for banks.[20] We consider the decentralized equilibrium without regulation in this setup. At the initial period, bank i will choose the amount to invest in the risky asset, n_i, the liquidity ratio, b_i, and the interest rate on the deposits, r_i, to maximize the net expected

[20] We restrict attention to deposit contracts that are in the form of simple debt contracts. Debt contracts can be justified by assuming that depositors can observe banks' asset returns only at a cost. According to Townsend (1979), in the case of costly state verification, debt contracts are optimal.

profits:

$$\max_{r_i,n_i,b_i} \quad (1-q)[(R+b_i)n_i - r_iL_i] + q\max\{R\gamma_i n_i - r_iL_i, 0\} - E - \Phi((1+b_i)n_i), \tag{18}$$

subject to

$$(1-q)r_iL_i + q\min\{R\gamma_i n_i,\ r_iL_i\} \geq L_i \qquad \textit{(IR)}, \tag{19}$$

where $\gamma_i = 1-(c-b_i)/P$ is the fraction of assets retained by banks at $t=1$ after fire sales, P is the price of risky assets (which, as before, banks take as given), and $L_i = (1+b_i)n_i - E$ is the amount of deposits. The bank has to satisfy the individual rationality (IR) constraint of consumers given in (19): The expected return to deposits must be greater than the initial deposit of a consumer, L_i. Each consumer receives a gross return of r_iL_i in the good state, which happens with probability $1-q$. In the bad state, which arises with probability q, the consumer obtains the minimum of the promised payment, r_iL_i, and the returns available to the bank after fire sales, $R\gamma_i n_i$. If $R\gamma_i n_i < r_iL_i$, the bank defaults in the bad state. We assume that a bank that is in default at $t=1$ is required by law to manage the remaining assets after fire sales until the final period and transfer generated resources, $R\gamma_i n_i$, to its depositors. However, we do not introduce any exogenous cost of bank default to its depositors or to the overall economy.

First, consider the choice of optimal r_i for a given investment level, n_i, and liquidity ratio, b_i. Because each bank is a local monopsony, the interest on deposit contracts needs to be just high enough to induce risk-neutral consumers to deposit their endowments with them. In technical terms, the individual rationality condition for consumers binds. If the returns in the bad state are such that $R\gamma_i n_i > L_i$, then banks can optimally set $r_i^* = 1$. In this case, similar to the basic model in Section 3, deposits are safe and consumers inelastically supply deposits to banks. However, if $R\gamma_i n_i < L_i$, banks have to offer a positive net interest rate to consumers in the good state to compensate for their losses in the bad state. For the IR condition of consumers to be satisfied, r_i has to be such that

$$r_i \geq \frac{L_i - qR\gamma_i n_i}{(1-q)L_i} \equiv r_i^*. \tag{20}$$

To obtain this constraint we rearrange the IR condition (19) and note that $\min\{R\gamma_i n_i, r_iL_i\} = R\gamma_i n_i$. In order to maximize profits, banks set $r_i = r_i^*$.[21] Now, we can substitute the optimal r_i^* back into the banks' objective function (18) and simplify to obtain

$$\max_{n_i,b_i} \quad (1-q)(R+b_i)n_i + qR\gamma_i n_i - (1+b_i)n_i - \Phi((1+b_i)n_i),$$

where we use $L_i + E = (1+b_i)n_i$ and $\max\{R\gamma_i n_i - r_iL_i, 0\} = 0$ because $R\gamma_i n_i - L_i < 0$ in equi-

[21] This setup requires the assumption that depositors can perfectly observe the equilibrium level of investment (n_i), the fraction of assets sold ($1-\gamma_i$), and the price of assets (P). However, we show in Section 5.2 that the results of the paper do not change when we change the environment by introducing deposit insurance, which does not require this perfect observation assumption.

librium, as argued earlier. Note that this problem is the same as the banks' problem that we considered in Section 3.2. Therefore, the liquidity ratio and level of risky investment in the decentralized equilibrium are the same as the benchmark model. We can easily show that the expected utility of a representative consumer is not affected by the current modification. After substituting (20) for the equilibrium interest rate, r_i^*, into the left-hand side of the IR condition (19), consumer's expected utility can be obtained as

$$(e - L_i) + (1-q)L_i \frac{L_i - qR\gamma_i n_i}{(1-q)Li} + qR\gamma_i n_i = e,$$

where we use that $\min\{R\gamma_i n_i, r_i L_i\} = R\gamma_i n_i$ in equilibrium. Therefore, we obtain that the social welfare function is also the same as the benchmark model. This implies that all results in the paper continue to hold in this setup.

5.2 Deposit insurance and limited liability

Deposit insurance can be introduced into the model with a slight modification. Suppose that the regulator (or a separate insurance agency) runs a domestic deposit insurance fund that is fairly priced. In particular, banks pay deposit insurance fees in good times and, in exchange, the deposit insurance covers any deficit between the banks' return and the promised payments to depositors in bad times. We assume that each bank is a local monopsony in the deposit market, as previously. Because of the deposit insurance, banks maximize profits by offering consumers a net zero interest rate. As a result, consumers inelastically supply their endowments to banks at the initial period. Let τ_i be the fee that banks pay to the deposit insurance in good times per unit of deposits. The banks' problem changes as follows:

$$\max_{n_i, b_i} \quad (1-q)[(R+b_i)n_i - L_i - \tau_i L_i] + q\max\{R\gamma_i n_i - L_i, 0\} - E - \Phi((1+b_i)n_i).$$

The fair pricing of the deposit insurance requires $(1-q)\tau_i L_i = q\ max\{L_i - R\gamma_i n_i, 0\}$. Substituting this fair value back into the banks' problem and noting that banks default in the bad state—that is, $max\{R\gamma_i n_i - L_i, 0\} = 0$—gives

$$\max_{n_i, b_i} \quad (1-q)(R+b_i)n_i + qR\gamma_i n_i - E - L_i - \Phi(n_i(1+b_i)).$$

Using $L_i = (1+b_i)n_i - E$, the last equation can be written as

$$\max_{n_i, b_i} \quad (1-q)(R+b_i)n_i + qR\gamma_i n_i - n_i(1+b_i) - \Phi(n_i(1+b_i)) \tag{21}$$

The problem of banks given by (21) is the same as in Section 3.2. Therefore, the liquidity ratio and optimal level of investment in the decentralized equilibrium without regulation remain the same. Note that the representative depositor consumes $e - L_i$ at the initial period and L_i in the final period in both states. As a result, the depositor's expected utility at $t = 0$ is $e - L_i + L_i = e$, and the social welfare function is also exactly the same as before. To conclude, all of the results in the paper are robust to adding a fairly priced deposit insurance and limited liability for banks in the model.

5.3 Operational costs of a bank

We utilize a convex operational cost for banks mainly for technical reasons. Without such a cost an equilibrium will not exist in general. However, the form of this function is not essential for our key results. In our model, the net interest rate on bank deposits is zero. Without an additional cost (such as an operational cost) banks can borrow more from depositors and park these funds as liquid assets (cash) in their portfolios. In that way they could freely insure against the fire sale risk. First, we believe that this is not realistic: Banks do face costs to attract deposits. Second, this setup completely ignores the opportunity cost of holding liquid assets, namely the cost of bygone profits from other investments. With few exceptions, most papers in the literature has a fixed balance sheet size which highlights this opportunity cost mechanism. However, with a fixed balance sheet size, the choice between risky assets and liquid assets boils down to a mere portfolio allocation problem. A setup with a single choice variable does not allow the type of interactions we study.

Furthermore, whether bank size matters for the inefficiencies banks create is also discussed in the context of the recent financial crisis, as well as how bank regulation might affect bank size. Regulatory rules might affect bank profitability, which may lead banks to resize their operations. To speak to these discussions, a flexible balance sheet size is important. Our result in Proposition 8 emphasizes that the composition of a bank's balance sheet matters more than its size, and that regulation does not necessarily imply a reduction in balance sheet size.

We choose a convex form similar to the ones imposed by Van den Heuvel (2008) and Acharya (2003, 2009), however our main results are not sensitive to the exact functional form. To be more specific, Propositions 1, 4 and 9 do not require any specific functional form and Proposition 6 is robust to some alternative modeling choices such as concave cost functions, like the square root or natural logarithm functions.[22]

5.4 Idiosyncratic liquidity shocks

In the basic model, the liquidity shock is aggregate in nature, as in Lorenzoni (2008). In this section, we show that the aggregate nature of the liquidity shock is without loss of generality and that our results do not change if we allow idiosyncratic liquidity shocks. In this more general setup, liquidity shocks hit only a fraction of the banks. Thus, banks are ex-post heterogeneous in terms of their liquidity needs. Banks that receive the liquidity shock need funding while others are left

[22]These results are not included here but are available upon request from the authors.

with excess liquidity. Banks with excess liquidity can use these resources to buy the risky assets from the distressed banks, potentially at fire sale prices.[23] Therefore, in this variant of the model, banks hoard liquidity also for a strategic purpose: They can use their liquid assets to buy risky assets at fire sale prices. This function of liquidity is also present in the models of Acharya, Shin, and Yorulmazer (2011), Allen and Gale (2004b), Allen and Gale (2004a), and Gorton and Huang (2004). The amount of risky assets that can be bought with the liquid holdings of a shock-free bank is $\frac{b_i n_i}{P}$.

First, we analyze the case conditionally on the liquidity shock but without knowing which banks receive the shock. We assume that, conditional on being in the bad state, the probability of being hit with a liquidity shock is λ for each bank. Hence, by the law of large numbers, a fraction λ of banks is hit by the liquidity shock in the bad state. The expected profit of a bank before the realization of which banks receive the shock, conditional on the bad state, is $\lambda R \gamma_i n_i + (1-\lambda)(n_i + \frac{b_i n_i}{P})R$. The first term, $\lambda R \gamma_i n_i$, is the return from remaining risky assets after fire sales multiplied by the probability of receiving the liquidity shock, λ. The amount of remaining risky assets after fire sales is denoted by γ_i for bank i, as in the benchmark model. The second term captures the returns from risky investment in the case without the liquidity shock, including the returns from risky asset bought using hoarded liquidity. We substitute for γ_i and rewrite the expected profit conditional on the bad state, as follows:

$$\begin{aligned}
\Pi_{i|bad} &= \lambda R \left(1 - \frac{c - b_i}{P}\right) n_i + (1-\lambda) n_i R + (1-\lambda)\frac{b_i n_i}{P} R, \\
&= \lambda R n_i - \frac{\lambda R c n_i}{P} + \lambda R \frac{b_i n_i}{P} + (1-\lambda) R n_i + \frac{b_i n_i}{P} R - \lambda \frac{b_i n_i}{P} R, \\
&= R n_i + R \left(\frac{b_i - c\lambda}{P}\right) n_i, \\
&= R \left(1 - \frac{c\lambda - b_i}{P}\right) n_i, \\
&= R \tilde{\gamma}_i n_i,
\end{aligned}$$

where $\tilde{\gamma}_i = 1 - \frac{c\lambda - b_i}{P}$. This $\tilde{\gamma}_i$ is similar to γ_i in the basic setup; the only difference is that the size of the liquidity shock, c, is replaced with $c\lambda$ in the numerator of the definition. In this setup, when we set $\lambda = 1$, we are back to our benchmark case. Thus, allowing λ to be between zero and one provides a more general model. In order to write the expected profits of banks at $t = 0$ in this more general setup, we simply note that the economy ends up in the bad state only with probability q and obtains the returns derived earlier, while good times arise with probability $1 - q$ and feature

[23] In principle, it is possible that the amount of excess liquidity in the banking system exceeds the liquidity needs of the shock-receiving banks. At the end of this subsection we explain why this situation does not arise in equilibrium.

returns that are the same as in the benchmark case:

$$\Pi_i = (1-q)(R+b_i)n_i + qR\tilde{\gamma}_i n_i,$$

where $\tilde{\gamma}_i = 1 - \frac{c\lambda - b_i}{P}$.

Compared with the benchmark case, the only difference in banks' expected profit at $t = 0$ is that c is replaced with $c\lambda$. For completeness, we conclude by writing the demand and supply functions in this more-general case. The aggregate liquidity need in the bad state is $\lambda(c-b)n$, and the liquidity supply is $(1-\lambda)bn + PQ^d(P)$. Equating demand and supply yields $\lambda(c-b)n = (1-\lambda)bn + PQ^d(P)$, and simplifying reduces this market-clearing condition to $(\lambda c - b)n = PQ^d(P)$. Compared with the market-clearing condition in the original setup, the only difference is, again, that c has been replaced with λc. Thus, in this new setup, if we relabel $\lambda c = \tilde{c}$, we are back to our original setup where c is replaced with $\tilde{c}$.

It would be possible to have no fire sales in the bad state in this setup if the liquid assets in the hands of shock-free banks were in excess of the liquidity need of shock-receiving banks, so that the risky assets were traded within the banking system without needing to sell to outside investors. Although this case is possible in principle, it is never observed in equilibrium because it is not optimal for banks to hoard sufficient liquidity for this case to arise. Comparing the demand for liquidity with the supply of liquidity in the case of the liquidity shock, it is clear that the fire sales arise if and only if λcn is greater than bn. In other words, fire sales are observed in equilibrium as long as $\tilde{c} > b$. Given that $\tilde{c}$ is a parameter, the ex-ante liquidity choice of banks determines whether fire sales occur. As we know from the benchmark case, banks optimally set $b_i < c$. Because this is true for any parameter value, it is true for $\tilde{c}$ as well. The intuition is the same: Holding liquidity is costly if the shock does not materialize. Thus, for banks to hoard liquidity, there must be some additional return to holding liquidity in case of the liquidity shock. This additional return is only possible if the fire sale price is less than R, which is only possible if there are fire sales. In other words, if there will not be any fire sales in the bad state—that is, if $P = R$—then there is no benefit to holding liquidity. But this contradicts the assumption of sufficient liquidity in the banking system.

6 Conclusion

In this paper, we investigate the optimal design of bank regulation and the interaction between capital and liquidity requirements. Our model is characterized by a fire sale externality, because atomistic banks do not take into account the effect of their initial portfolio choices on the fire sale price. Existence of this fire sale externality creates an inefficiency. In the unregulated competitive equilibrium, banks overinvest in the risky asset and underinvest in the liquid asset compared to a constrained planner's allocations. We investigate whether the constrained efficient allocations can be implemented using quantity-based capital and liquidity regulations, as in the Basel Accords. The regulation required is macroprudential because it addresses the instability in the banking system by targeting aggregate capital and liquidity ratios.

Our results indicate that the pre-Basel III regulatory framework, with its reliance only on capital requirements, was inefficient and ineffective in addressing systemic instability caused by fire sales. Capital requirements can lead to less severe fire sales by forcing banks to reduce risky assets—however, we show that banks respond to stricter capital requirements by decreasing their liquidity ratios. Anticipating this response, the regulator preemptively sets capital ratios at high levels. Ultimately, this interplay between banks and the regulator leads to inefficiently low levels of risky assets and liquidity. Macroprudential liquidity requirements that complement capital regulations, as in Basel III, restore constrained efficiency, improve financial stability and allow for a higher level of investment in risky assets.

It is important to highlight that our results cannot be interpreted as indicating that the actual capital regulation requirements were too high in a particular country (such as the U.S.) in the pre-crisis period, which corresponds to pre-Basel III framework, and that now they should be relaxed. Our results only say that if capital regulations were set optimally from a welfare maximizing point in the absence of liquidity regulation, they would be set at inefficiently high levels compared to the second-best environment in which the regulator is also endowed with the liquidity regulation tool. Our model is not meant to be quantitative and hence does not speak to whether actual capital ratios in practice either under Basel I/II or Basel III are too low or too high. However, many studies, most famously Admati et al. (2010), have argued that current minimum capital requirements are too low.

The message of this paper goes beyond bank regulation. Our results imply that capital ratios are not a good predictor of the stability of the banking system or any individual bank under a potential distress scenario. Without sufficient liquidity buffers, banks' capital can easily erode with fire sale losses. Under fire sale externalities, then, a well-capitalized banking system may experience greater losses than a less-capitalized banking system with strong liquidity buffers. Thus, capital ratios alone cannot be barometers of soundness of individual banks or a banking system.

The Basel III liquidity ratio LCR currently applies to only large banks in the U.S. In contrast, our results suggests that liquidity regulations should apply even to small banks because in our model all banks are small by definition, as we consider atomistic banks that engage in fire sales markets and take asset prices as given. Answering the question of whether liquidity regulations should be applied differently to large and small banks, like the question of whether they should be applied differently to well-capitalized and poorly-capitalized banks, is beyond the scope of our current model. We leave these interesting theoretical and policy questions to future research.

References

Acharya, V. (2003). Is the international convergence of capital adequacy regulation desirable? *The Journal of Finance 58*(6), 2745–2782.

Acharya, V. (2009). A theory of systemic risk and design of prudential bank regulation. *Journal of Financial Stability 5*(3), 224–255.

Acharya, V. and T. Yorulmazer (2008). Cash-in-the-market pricing and optimal resolution of bank failures. *Review of Financial Studies 21*(6), 2705–2742.

Acharya, V. V., H. Mehran, and A. Thakor (2015). Caught between scylla and charybdis? Regulating bank leverage when there is rent seeking and risk shifting. *Review of Corporate Finance Studies* (Forthcoming).

Acharya, V. V., H. S. Shin, and T. Yorulmazer (2011). Crisis resolution and bank liquidity. *Review of Financial studies 24*(6), 2166–2205.

Admati, A. R., P. M. DeMarzo, M. F. Hellwig, and P. C. Pfleiderer (2010). Fallacies, irrelevant facts, and myths in the discussion of capital regulation: Why bank equity is not expensive. *Stanford Graduate School of Business Research Paper 2065.*

Allen, F. (2014). How should bank liquidity be regulated? Manuscript.

Allen, F. and D. Gale (1994). Limited market participation and volatility of asset prices. *The American Economic Review 84*(4), 933–955.

Allen, F. and D. Gale (1998). Optimal financial crises. *The Journal of Finance 53*(4), 1245–1284.

Allen, F. and D. Gale (2004a). Financial fragility, liquidity, and asset prices. *Journal of the European Economic Association 2*(6), 1015–1048.

Allen, F. and D. Gale (2004b). Financial intermediaries and markets. *Econometrica 72*(4), 1023–1061.

Allen, F. and D. Gale (2005). From cash-in-the-market pricing to financial fragility. *Journal of the European Economic Association 3*(2-3), 535–546.

Almazan, A. (2002). A model of competition in banking: Bank capital vs expertise. *Journal of Financial Intermediation 11*(1), 87–121.

Amir, R. (1996). Cournot oligopoly and the theory of supermodular games. *Games and Economic Behavior 15*(2), 132–148.

Arseneau, D. M., D. E. Rappoport, and A. P. Vardoulakis (2015). Secondary market liquidity and the optimal capital structure. *Finance and Economics Discussion Series, Board of Governors of the Federal Reserve System* (No. 2015-31).

Bernanke, B. and M. Gertler (1989). Agency costs, net worth, and business fluctuations. *The American Economic Review 79*(1), 14–31.

Bhattacharya, S. and D. Gale (1987). Preference shocks, liquidity, and central bank policy. In W. Barnett and K. Singleton (Eds.), *New Approaches to Monetary Economics*, pp. 69–88. Cambridge University Press.

Borch, K. (1962). Equilibrium in a reinsurance market. *Econometrica 30*(3), 424–444.

Bouwman, C. H. (2012). Liquidity: How banks create it and how it should be regulated. In A. N. Berger, P. Molyneux, and J. O. S. Wilson (Eds.), *The Oxford Handbook of Banking, 2nd edition*, pp. 182. Oxford University Press.

Brunnermeier, M. K. and L. H. Pedersen (2009). Market liquidity and funding liquidity. *Review of Financial Studies 22*(6), 2201–2238.

Calomiris, C., F. Heider, and M. Hoerova (2013). A theory of bank liquidity requirements. Working Paper.

Coval, J. and E. Stafford (2007). Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics 86*(2), 479–512.

Covas, F. and J. C. Driscoll (2014). Bank liquidity and capital regulation in general equilibrium. *Finance and Economics Discussion Series, Board of Governors of the Federal Reserve System* (No. 2014-85).

Crandall, R. W. and J. D. Graham (1984). Automobile safety regulation and offsetting behavior: Some new empirical estimates. *The American Economic Review 74*(2), 328–331.

De Nicoló, G., A. Gamba, and M. Lucchetta (2012). Capital regulation, liquidity requirements and taxation in a dynamic model of banking. *IMF Working Papers* (12-72).

Dell'Ariccia, G. and R. Marquez (2006). Competition among regulators and credit market integration. *Journal of Financial Economics 79*(2), 401–430.

Diamond, D. W. and P. H. Dybvig (1983). Bank runs, deposit insurance, and liquidity. *The Journal of Political Economy*, 401–419.

Diamond, D. W. and R. G. Rajan (2011). Fear of fire sales, illiquidity seeking, and credit freezes. *The Quarterly Journal of Economics 126*(2), 557–591.

Farhi, E., M. Golosov, and A. Tsyvinski (2009). A theory of liquidity and regulation of financial intermediation. *The Review of Economic Studies 76*(3), 973–992.

Fisher, I. (1933). The debt-deflation theory of great depressions. *Econometrica 1*(4), 337–357.

Freixas, X. and J.-C. Rochet (2008). *Microeconomics of Banking.* The MIT Press.

Gai, P., S. Kapadia, S. Millard, and A. Perez (2008). Financial innovation, macroeconomic stability and systemic crises. *The Economic Journal 118*(527), 401–426.

Geanakoplos, J. and H. M. Polemarchakis (1986). Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete. In W. P. Heller, R. M. Starr, and D. A. Starrett (Eds.), *Uncertainty, information and communication: Essays in honor of Kenneth J. Arrow*, Volume 3, pp. 65–96. Cambridge University Press Cambridge, UK.

Gorton, G. and L. Huang (2004). Liquidity, efficiency, and bank bailouts. *The American Economic Review 94*(3), 455–483.

Greenwald, B. C. and J. E. Stiglitz (1986). Externalities in economies with imperfect information and incomplete markets. *The Quarterly Journal of Economics 101*(2), 229–264.

Hart, O. D. (1975). On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory 11*(3), 418–443.

Hellmann, T., K. Murdock, and J. Stiglitz (2000). Liberalization, moral hazard in banking, and prudential regulation: Are capital requirements enough? *American Economic Review 90*(1), 147–165.

Holmström, B. and J. Tirole (2001). Lapm: A liquidity-based asset pricing model. *the Journal of Finance 56*(5), 1837–1867.

Kara, G. I. (2015). Systemic risk, international regulation, and the limits of coordination. *Journal of International Economics* (http://dx.doi.org/10.1016/j.jinteco.2015.11.007).

Kashyap, A. K., D. P. Tsomocos, and A. P. Vardoulakis (2014). How does macroprudential regulation change bank credit supply? *NBER Working Paper* (20165).

Kiyotaki, N. and J. Moore (1997). Credit cycles. *The Journal of Political Economy 105*(2), 211–248.

Korinek, A. (2011). Systemic risk-taking: Amplification effects, externalities, and regulatory responses. *ECB Working Paper Series* (No. 1345).

Krishnamurthy, A. (2003). Collateral constraints and the amplification mechanism. *Journal of Economic Theory 111*(2), 277–292.

Krishnamurthy, A. (2010). Amplification mechanisms in liquidity crises. *American Economic Journal: Macroeconomics 2*(3), 1–30.

Lipsey, R. G. and K. Lancaster (1956). The general theory of second best. *The Review of Economic Studies 24*(1), 11–32.

Lorenzoni, G. (2008). Inefficient credit booms. *The Review of Economic Studies 75*(3), 809–833.

Minsky, H. P. (1992). The financial instability hypothesis. *The Jerome Levy Economics Institute Working Paper* (74).

Peltzman, S. (1975). The effects of automobile safety regulation. *The Journal of Political Economy*, 677–725.

Perotti, E. and J. Suarez (2011). A pigovian approach to liquidity regulation. *International Journal of Central Banking 7*(4).

Pulvino, T. (2002). Do asset fire sales exist? An empirical investigation of commercial aircraft transactions. *The Journal of Finance 53*(3), 939–978.

Repullo, R. (2005). Liquidity, risk taking, and the lender of last resort. *International Journal of Central Banking 1*(2).

Rochet, J.-C. (2008). Liquidity regulation and the lender of last resort. *Financial Stability Review 11*, 45–52.

Shleifer, A. and R. Vishny (1992). Liquidation values and debt capacity: A market equilibrium approach. *The Journal of Finance 47*(4), 1343–1366.

Stein, J. C. (2012). Monetary policy as financial stability regulation. *The Quarterly Journal of Economics 127*(1), 57–95.

Stein, J. C. (2013). Liquidity regulation and central banking. In *Speech at the" Finding the Right Balance" 2013 Credit Markets Symposium sponsored by the Federal Reserve Bank of Richmond, Charlotte, North Carolina.*

Stiglitz, J. E. (1982). The inefficiency of the stock market equilibrium. *The Review of Economic Studies 49*(2), 241–261.

Tarullo, D. K. (2014). Liquidity regulation. In *Speech at the 2014 Clearing House Annual Conference. New York, NY.*

Tirole, J. (2011). Illiquidity and all its friends. *Journal of Economic Literature*, 287–325.

Townsend, R. (1979). Optimal contracts and competitive markets with costly state verification. *Journal of Economic Theory 21*(2), 265–93.

Uhlig, H. (2010). A model of a systemic bank run. *Journal of Monetary Economics 57*(1), 78–96.

Van den Heuvel, S. J. (2008). The welfare cost of bank capital requirements. *Journal of Monetary Economics 55*(2), 298–320.

Vives, X. (2001). *Oligopoly pricing: old ideas and new tools.* MIT press.

Walther, A. (2015). Jointly optimal regulation of bank capital and maturity structure. *Journal of Money, Credit, and Banking* (Forthcoming).

Williamson, O. (1988). Corporate finance and corporate governance. *The Journal of Finance 43*(3), 567–591.

7 Appendix

7.1 A closed-form solution for the competitive equilibrium

Let outside investors' technology function be given by $F = Rln(1+y)$. Outside investors choose how much assets, y, to buy from banks in the bad state at $t = 1$ to maximize their profits, $F(y) - Py$, where P is the price of assets. The first-order condition of this problem yields (inverse) demand function of outside investors for risky assets:

$$P = F'(y) = \frac{R}{1+y} \text{ and hence } y = F'^{-1}(P) = \frac{R}{P} - 1 \equiv Q^d(P). \tag{22}$$

We solve for the competitive equilibrium price, P, in the main text, as shown by (9). Now, use this solution in the demand side function and define the total amount of assets purchased by outside investors, τ, in terms of the exogenous variables as follows:

$$y = \frac{R}{P} - 1 = \frac{R - 1 + q}{q(1+c)} - 1 \equiv \tau. \tag{23}$$

We obtain the total supply of asset by banks as $(1-\gamma)n$ by (4) in Section 3.1.2. Hence, the market clearing condition, $(1-\gamma)n = \tau$, yields:

$$(c - b)n = P\tau \implies n = \frac{P\tau}{c - b}. \tag{24}$$

This equation gives the investment level, n, as a function of the liquidity ratio, b. We can solve for the latter from the first-order conditions of banks' problem in the decentralized case, given by (46-47), as derived in the proof of Proposition 2 below. Using $\frac{R}{P} = \tau + 1$ from (23) and the functional form of the operational cost, $\Phi'(n(1+b)) = 2dn(1+b)$, in the first order condition with respect to b, given by (47) yields:

$$\begin{aligned} 1 - q + q(\tau + 1) &= 1 + 2dn(1+b), \\ 1 + q\tau &= 1 + 2d\frac{P\tau}{c-b}(1+b), \end{aligned}$$

where in the second line we use $n = P\tau/(c-b)$ from (24). Substituting $R/(\tau+1)$ for P from (23) yields

$$c - b = \frac{2d}{q}\frac{R(1+b)}{\tau+1}.$$

Finally, rearrange to obtain the liquidity ratio in the competitive equilibrium as

$$b = \frac{cq(\tau+1) - 2dR}{q(\tau+1) + 2dR}. \tag{25}$$

To obtain the risky investment level in the competitive equilibrium substitute this expression for b in (24):

$$n = \frac{\tau}{\tau+1}\frac{q(\tau+1) + 2dR}{2d(1+c)} \tag{26}$$

7.2 A closed-form solution for the constrained planner's problem

Proposition 4 allows us to focus on the case $b < c$ when analyzing the constrained planner's problem. The planner chooses $n, b \geq 0$ to solve:

$$\max_{n,b} W(n,b) = \; (1-q)\{R+b\}n + qR\gamma n - D(n(1+b)) + q[F((1-\gamma)n) - P(1-\gamma)n] + e,$$

subject to the society's budget constraint at $t = 0$, $0 \leq (1+b)n \leq e + E$. The term in brackets, $F((1-\gamma)n) - P(1-\gamma)n$, gives the profits of outside investors' in the bad state. After substituting $P(1-\gamma)n = (c-b)n$ from the market clearing condition, the first order conditions of the planner's problem with respect to n and b are respectively:

$$\begin{aligned}(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial\gamma}{\partial n}n\Big\} + q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right) - c + b\right\} &= D^{'}(n(1+b))(1+b), \\ (1-q)n + qR\frac{\partial\gamma}{\partial b}n + q\left\{F'((1-\gamma)n)\left(-\frac{\partial\gamma}{\partial b}\right)n + n\right\} &= D^{'}(n(1+b))n, \qquad (27)\end{aligned}$$

where $\gamma = 1 + \frac{b-c}{P}$ from banks' problem in the bad state, as obtained in Section 3.1.2. Combining the two first-order conditions to obtain:

$$\begin{aligned}(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial\gamma}{\partial n}n\Big\} + q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right) - c + b\right\} = \\ \left[(1-q) + qR\frac{\partial\gamma}{\partial b} + qF'((1-\gamma)n)\left(-\frac{\partial\gamma}{\partial b}\right) + q\right](1+b) \qquad (28)\end{aligned}$$

First, note that using the functional form for outside investors' demand, given by (22), in the market clearing condition (5) yields the price of assets in the bad state as a function of initial portfolio allocations:

$$E(P,n,b) = Q^d(P) - Q^s(P,n,b) = 0 \implies \frac{R-P}{P} = \frac{c-b}{P}n \implies P = R - (c-b)n. \qquad (29)$$

Substituting $\frac{\partial\gamma}{\partial n} = -\frac{(c-b)^2}{P^2}$ and $\frac{\partial\gamma}{\partial b} = \frac{R}{P^2}$, and later $P = R - (c-b)n$ into (28) and noting that $F'((1-\gamma)n) = P$ yields:

$$\begin{aligned}(1-q)(R+b) + qR\Big\{1 - \frac{c-b}{P} - \frac{(c-b)^2}{P^2}n\Big\} + q\left\{P\left(\frac{c-b}{P} + \frac{(c-b)^2}{P^2}n\right) - c + b\right\} = \\ \left[(1-q) + qR\frac{R}{P^2} - qP\frac{R}{P^2} + q\right](1+b),\end{aligned}$$

or equivalently:

$$\begin{aligned}(1-q)(R-1) + (1-q)(1+b) + qR - qR\Big\{\frac{(c-b)P + (c-b)^2 n}{P^2}\Big\} + q\left\{P\frac{(c-b)P + (c-b)^2 n}{P^2} - c + b\right\} = \\ (1-q)(1+b) + q\frac{R}{P^2}(R-P)(1+b) + q(1+b).\end{aligned}$$

Note that $(c-b)P+(c-b)^2n = (c-b)[R-(c-b)n]+(c-b)^2n = R(c-b)$. Substitute this equilavance into the equation above and simplify:

$$R-1+q-qR\frac{R(c-b)}{P^2}+qP\frac{R(c-b)}{P^2}-qc+qb = q\frac{R}{P^2}(R-P)(1+b)+q+qb.$$

Further simplification yields:

$$\begin{aligned}
R-1-qc &= \frac{qR}{P^2}\{(R-P)(1+b)+R(c-b)-P(c-b)\} \\
R-1-qc &= \frac{qR}{P^2}\{(R-[R-(c-b)n](1+b)+R(c-b)-[R-(c-b)n](c-b)\} \\
R-1-qc &= \frac{qR}{P^2}\{(R-R+(c-b)n(1+b)+R(c-b)-R(c-b)+(c-b)^2n\} \\
R-1-qc &= \frac{qR}{P^2}\{(c-b)n(1+b)+(c-b)^2n\} \\
R-1-qc &= \frac{qR}{P^2}\{(c-b)n(1+b+c-b)\} \\
R-1-qc &= \frac{qR(c-b)n(1+c)}{P^2} \\
R-1-qc &= \frac{qR(R-P)(1+c)}{P^2}, \qquad (30)
\end{aligned}$$

where we substitute $P = R-(c-b)n$ in the second line using the market clearing condition (29), and $(c-b)n = R-P$ using the same condition again in the last line above. From (30) we obtain the following quadratic equation in P:

$$(R-1-qc)P^2+qR(1+c)P-qR^2(1+c) = 0, \qquad (31)$$

which we can solve for the price of assets under constrained planner's solution, P^{**}:

$$P^{**} = \frac{-qR(1+c)+\sqrt{q^2R^2(1+c)^2+4(R-1-qc)qR^2(1+c)}}{2(R-1-qc)}.$$

We can define $\tau^{**} \equiv R/P^{**}-1$ similar to (23) to represent the total amount of assets sold under fire sales to outside investors in terms of the model parameters, and write risky investment as a function of the liquidity ratio as $n^{**} = P^{**}\tau^{**}/(c-b)$ using the market clearing condition, similar to (29).

We use these equations to solve for the constrained efficient portfolio allocations n^{**}, b^{**}. For that start from the first order condition with respect to b given above by (27):

$$\begin{aligned}
1-q+qR\frac{\partial\gamma}{\partial b}+q\left\{F'((1-\gamma)n)\left(-\frac{\partial\gamma}{\partial b}\right)+1\right\} &= D^{'}(n(1+b)), \\
1-q+qR\frac{R}{P^2}+q\left\{-P\frac{R}{P^2}+1\right\} &= 1+2dn(1+b), \\
q\frac{R^2}{P^2}-q\frac{R}{P} &= 2dn(1+b).
\end{aligned}$$

Writing all endogenous variables in terms of τ^* and simplifying yields

$$\begin{aligned}
q(\tau^{**}+1)^2 - q(\tau^{**}+1) &= 2d\frac{P\tau^{**}}{c-b}(1+b), \\
q(\tau^{**}+1)(\tau^{**}+1-1) &= 2d\frac{R}{\tau^{**}+1}\frac{\tau^{**}}{c-b}(1+b), \\
q(\tau^{**}+1)^2\tau^{**}(c-b) &= 2dR\tau^{**}(1+b), \\
q(\tau^{**}+1)^2 c - 2dR &= b\{2dR + q(\tau^{**}+1)^2\},
\end{aligned}$$

where we use $R/P^{**} = \tau^{**}+1$ and $n^{**} = P^{**}\tau^{**}/(c-b)$. For future reference, using the second from the last number, we can obtain the liquidity shortage per risky asset in the constrained planner's solution as

$$c - b^{**} = \frac{2dR(1+b^{**})}{q(\tau^{**}+1)^2}.$$

We can obtain the closed-form solution for the constrained efficient liquidity ratio, b^{**}, by rearranging the last equation above, as

$$b^{**} = \frac{cq(\tau^{**}+1)^2 - 2dR}{q(\tau^{**}+1)^2 + 2dR}. \tag{32}$$

Finally, we can obtain the closed-form solution for the risky investment level by substituting b^{**} into $n^{**} = P^{**}\tau^{**}/(c-b)$ and using $P^{**} = R/(\tau^{**}+1)$

$$\begin{aligned}
n^{**} &= \frac{P\tau^{**}}{c-b}, \\
&= \frac{R\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2 + 2dR}{2dR(1+c)}, \\
&= \frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2 + 2dR}{2d(1+c)}.
\end{aligned} \tag{33}$$

7.3 A closed-form solution for the partial regulation case

In the partial regulation case, we consider the problem of a regulator who chooses the optimal level of risky investment, $n \geq 0$, at $t = 0$ to maximize the net expected social welfare but who allows banks to freely choose their liquidity ratio, b_i. The bank chooses the liquidity ratio, b_i, to maximize its expected profits; hence, the problem of the bank is as follows:

$$\max_{b_i} \Pi_i(b_i; n) = (1-q)\{R + b_i\}n + qR\gamma_i n - D(n(1+b_i)).$$

The first-order condition of the banks' problem (7.3) with respect to b_i is

$$1 - q + qR\frac{1}{P} = D^{'}(n(1+b_i)). \tag{34}$$

We use the same functional form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section 7.1 and constrained planner's problem in Section 7.2. We can also define $\tau^* \equiv R/P^* - 1$ similar to (23) to represent the total amount of assets sold under fire sales to outside investors in terms of the model parameters, and write risky investment as a function of the liquidity ratio as $n^* = P^*\tau^*/(c-b)$ using the market clearing condition, similar to (29). Now, use the functional-form for the operational cost in banks' first-order condition and manipulate

$$\begin{aligned} 1 - q + \frac{qR}{P} &= 1 + 2dn(1+b), \\ q\left(\frac{R}{P} - 1\right) &= 2d\frac{P\tau}{c-b}(1+b), \\ q\tau &= 2d\frac{R}{\tau+1}\frac{\tau}{c-b}(1+b), \end{aligned}$$

where we first use $n = \frac{P\tau}{c-b}$ and then substitute $P = \frac{R}{\tau+1}$. From the last equation we can obtain an expression for the liquidity ratio in this case in terms of τ^* as follows

$$b^* = \frac{qc(\tau^*+1) - 2dR}{q(\tau^*+1) + 2dR}. \tag{35}$$

Using $n = \frac{P\tau}{c-b}$ and $P = \frac{R}{\tau+1}$ once more, we can obtain a similar expression for the risky investment level in this case in terms of τ^* as follows:

$$n^* = \frac{\tau^*}{\tau^*+1}\frac{q(\tau^*+1) + 2dR}{2d(1+c)}. \tag{36}$$

All that remains now is to obtain a closed-form solution for $\tau^* = R/P^* - 1$, and substitute that in (35) and (36) to obtain closed-form solutions for n^* and b^*. To obtain a closed-form solution for P^* we analyze the regulator's problem. The regulator takes into account that for any given n, the banks optimally choose their liquidity ratio $b(n)$, as shown by the response function (14). Hence, we can write the regulator's objective function as:

$$\max_n \; W(n) = (1-q)\{R + b(n)\}n + qR\gamma n + q[F((1-\gamma)n) - (c - b(n))n] - D((1+b(n))n),$$

from which we can obtain the following first order conditions with respect to n as

$$(1-q)\{R + b(n) + nb^{'}(n)\} + qR\left\{\gamma + n\frac{d\gamma}{dn}\right\} + q[F'(\cdot)\left(1 - \gamma - \frac{d\gamma}{dn}n\right) - c + b(n) + nb'(n)] = D^{'}(n(1+b))\{1 + b(n) + nb^{'}(n)\}. \tag{37}$$

We use the same functional-form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section 7.1 and constrained planner's problem in Section 7.2. First, note that substituting for P using (29) into γ, given by (3), we get

$$\gamma = 1 + \frac{b(n) - c}{P} = 1 + \frac{b(n) - c}{R + (b(n) - c)n},$$

Using this equivalence, we can obtain the total derivative of γ with respect to n as:

$$\begin{aligned}\frac{d\gamma}{dn} &= \frac{\partial\gamma}{\partial b}b'(n)+\frac{\partial\gamma}{\partial n}\\ &= \frac{P-(b(n)-c)n}{P^2}b'(n)-\frac{(b(n)-c)^2}{P^2}\\ &= \frac{b'(n)}{P}-\frac{nb'(n)(b(n)-c)}{P^2}-\frac{(b(n)-c)^2}{P^2}. \end{aligned} \tag{38}$$

Replacing $d\gamma/dn$ in the first-order condition (37) with (38) and rearranging yields

$$\begin{aligned}&(1-q)\{R+b(n)\}+qR\left(1+\frac{b(n)-c}{P}-\frac{n(b(n)-c)^2}{P^2}\right)+nb^{'}(n)\left\{1-q+\frac{qR}{P}-D'(\cdot)-\frac{qR(b(n)-c)n}{P^2}\right\}\\ &\quad+q\left[\left(-\frac{b(n)-c}{P}-\frac{b'(n)n}{P}+\frac{n^2b'(n)(b(n)-c)}{P^2}+\frac{n(b(n)-c)^2}{P^2}\right)P+(b(n)-c)+nb'(n)\right]\\ &\quad-D^{'}(\cdot)\{1+b(n)\}=0,\end{aligned}$$

where we replace $F'((1-\gamma)n)=P$ using the market clearing condition (29) in the second line. We have that $1-q+qR/P-D'(\cdot)=0$ from the banks' first-order condition (34). Hence, the first-order condition above can further be simplified as follows:

$$\begin{aligned}&R-1+q-\frac{qR^2(1+c)}{P^2}-\frac{qRn(b(n)-c)(1+b(n))}{P^2}-\frac{qRb'(n)n^2(b(n)-c)}{P^2}\\ &\quad+\frac{qn(b(n)-c)^2P}{P^2}+\frac{qb'(n)n^2(b(n)-c)P}{P^2}=0.\end{aligned}$$

Divide the last equation by qR to obtain

$$\begin{aligned}&\frac{R-1+q}{qR}-\frac{R(1+c)}{P^2}-\frac{n(b(n)-c)(1+b(n))}{P^2}-\frac{b'(n)n^2(b(n)-c)}{P^2}\\ &\quad+\frac{n(b(n)-c)^2P}{RP^2}+\frac{b'(n)n^2(b(n)-c)P}{RP^2}=0.\end{aligned}$$

Let us define

$$\sigma\equiv\frac{R-1+q}{qR}. \tag{39}$$

Using this definition, we can write this first-order condition as

$$\begin{aligned}&\frac{1}{P^2}\left\{\sigma P^2-R(1+c)-n(b(n)-c)(1+b(n))-b^{'}(n)(b(n)-c)n^2\right\}\\ &\quad+\frac{1}{P^2}\left\{\frac{(b-c)^2nP}{R}+b^{'}(n)\frac{n^2(b-c)P}{R}\right\}=0.\end{aligned} \tag{40}$$

We focus on the terms inside the braces because in equilibrium price must be strictly positive. Using this term, we would like to write endogenous variables n and b in terms of the parameters of the model and P, and then, use these expression in the first-order conditions of the banks' problem (34) to obtain a closed-form solution for P. For that, first, below we obtain $1+b(n)$, $n(b(n)-c)$ and

$b'(n)$ in terms of the parameters of the model and P starting from the banks' first-order condition (34):

$$
\begin{aligned}
(1-q)+q\frac{R}{P} &= 1+2dn(1+b), \qquad (41)\\
q(R-P) &= P2dn(1+b),\\
q(R-P) &= [R+(b-c)n]2dn(1+b),\\
-q(b-c)n &= 2dn(1+b)R+2dn(1+b)(b-c)n,\\
-(b-c)[q+2dn(1+b)] &= 2d(1+b)R,
\end{aligned}
$$

where we substitute for $P = R+(b-c)n$ using (29). Now, take the derivative of both sides with respect to n, and collect terms that involve $b^{'}(n)$:

$$
\begin{aligned}
-b'(n)[q+2dn(1+b)]-2d(b-c)[1+b+nb'(n)] &= 2dRb'(n),\\
-b'(n)[q+2dn(1+b)]-2d(b-c)(1+b)-2d(b-c)nb'(n) &= 2dRb'(n),\\
-b'(n)[2dR+2dn(b-c)+q+2dn(1+b)] &= 2d(b-c)(1+b),\\
-b'(n)[2dR+q+2dn(2b+1-c)] &= 2d(b-c)(1+b).
\end{aligned}
$$

From the last equation we obtain:

$$
b'(n) = \frac{-2d(b-c)(1+b)}{2dR+q+2dn(2b+1-c)}. \qquad (42)
$$

We further simplify $b'(n)$ in order to eliminate b from this expression. In order to do this simplification, note that first, from the market clearing condition at $t = 1$, $P = R+(b-c)n$, as derived in (29), we can obtain that

$$
b-c = -\frac{R-P}{n}.
$$

Second, from the banks' first-order condition, given by (41), we can obtain that

$$
1+b = \frac{q}{2dn}\left(\frac{R}{P}-1\right).
$$

Use these values for $1+b$ and $b-c$ into (42) to write $b'(n)$ as a function of n, P and the parameters of the model as follows

$$
\begin{aligned}
b'(n) &= \frac{-2d(-1)\frac{R-P}{n}\frac{q}{2dn}\left(\frac{R}{P}-1\right)}{2dR+q-2d(R-P)+2d\frac{q}{2d}\left(\frac{R}{P}-1\right)},\\
&= \frac{\frac{q}{n^2P}(R-P)^2}{\frac{1}{P}[2dRP+qP-2dP(R-P)+q(R-P)]},\\
&= \frac{q(R-P)^2}{n^2[2dRP+qP-2dRP+2dP^2+qR-qP]},\\
&= \frac{q(R-P)^2}{n^2[2dP^2+qR]}.
\end{aligned}
$$

Eventually, use the expressions obtained for $1+b(n)$, $n(b(n)-c)$ and $b'(n)$ above to rewrite the term inside the braces in (40) as:

$$\sigma P^2 - R(1+c) + (R-P)\frac{q(R-P)}{2dPn} + \frac{q(R-P)^2}{n^2[2dP^2+qR]}\frac{R-P}{n}n^2 + \frac{P(R-P)^2}{nR} - \frac{q(R-P)^2}{n^2[2dP^2+qR]}\frac{R-P}{nR}Pn^2 = 0$$

$$\sigma P^2 - R(1+c) + \frac{q(R-P)^2}{n}\left[\frac{1}{2dP} + \frac{R-P}{2dP^2+qR}\right] + \frac{(R-P)^2 P}{nR}\frac{2dP^2+qP}{2dP^2+qR} = 0$$

Note that the last equation takes the form of $A + B/n + C/n = 0$ where A, B, C group relevant terms. Therefore, we can obtain n in the form of $n = -B/A - C/A$, that is, from the last equation we can obtain n in terms of P and the parameters of the model:

$$n = \frac{q(R-P)^2\left[\frac{1}{2dP} + \frac{R-P}{2dP^2+qR}\right]}{R(1+c)-\sigma P^2} + \frac{(R-P)^2\frac{(2dP^2+qP)P}{(2dP^2+qR)R}}{R(1+c)-\sigma P^2} \equiv \psi(P) + \phi(P).$$

We can similarly obtain an expression for b in terms of P and the parameters of the model using the equilibrium price function $P = R + (b-c)n$, which implies that

$$b = \frac{P-R}{n} + c = \frac{P-R+cn}{n} = \frac{P-R+c[\psi(P)+\phi(P)]}{\psi(P)+\phi(P)}.$$

Now, substitute these expressions for n and b back into the banks' first-order condition (41) in order to obtain a fixed-point equation that involves only P as an endogenous variable, from which we can solve for the equilibrium price P:

$$\begin{aligned}
2dn(1+b) &= -q + \frac{qR}{P}, \\
2d[\psi(P)+\phi(P)]\left[\frac{P-R+c[\psi(P)+\phi(P)]}{\psi(P)+\phi(P)} + 1\right] + q &= \frac{qR}{P} \\
2dP\{P-R+(1+c)[\psi(P)+\phi(P)]\} + qP &= qR \\
-2dP(R-P) + 2dP(1+c)[\psi(P)+\phi(P)]\} &= q(R-P) \\
2d(1+c)P[\psi(P)+\phi(P)] &= (2dP+q)(R-P) \\
2d(1+c)P(R-P)^2\left\{q\left[\frac{1}{2dP} + \frac{R-P}{2dP^2+qR}\right] + \frac{P(2dP^2+qP)}{R(2dP^2+qR)}\right\} &= [R(1+c)-\sigma P^2](R-P)(2dP+q) \\
2d(1+c)P(R-P)\left\{q\frac{R(2dPR+qR)}{(2dP^2+qR)R} + \frac{2dP^2(2dP^2+qP)}{2dPR(2dP^2+qR)}\right\} &= [R(1+c)-\sigma P^2](2dP+q)
\end{aligned}$$

Now, we sum the terms in side the braces on the left-hand side and multiply both sides with the common denominator of the left-hand side after summation and simplify further to get:

$$\begin{aligned}
2d(1+c)P(R-P)\{qR(2dPR+qR)+2dP^2(2dP^2+qP)\} &= [R(1+c)-\sigma P^2](2dP+q)2dPR(2dP^2+qR) \\
(1+c)(R-P)\{qR^2(2dP+q)+2dP^3(2dP+q)\} &= [R(1+c)-\sigma P^2](2dP+q)R(2dP^2+qR) \\
(1+c)(R-P)(2dP+q)(qR^2+2dP^3) &= [R(1+c)-\sigma P^2](2dP+q)R(2dP^2+qR) \\
(1+c)(R-P)(qR^2+2dP^3) &= [R(1+c)-\sigma P^2]R(2dP^2+qR) \\
(1+c)R(qR^2+2dP^3)-(1+c)P(qR^2+2dP^3) &= R^2(1+c)(2dP^2+qR)-\sigma P^2R(2dP^2+qR) \\
(1+c)R2dP^2-(1+c)qR^2-(1+c)2dP^3 &= R^2(1+c)2dP-\sigma R2dP^3-\sigma PRqR
\end{aligned}$$

We can rearrange this last equation to obtain a cubic equation in terms of the partial equilibrium price:

$$\begin{aligned}
(\sigma 2dR-(1+c)2d)P^3+(1+c)R2dP^2+[\sigma qR^2-R^2(1+c)2d]P-(1+c)qR^2 &= 0 \\
2d(\sigma R-1-c)P^3+2dR(1+c)P^2+R^2(\sigma q-2d(1+c))P-(1+c)qR^2 &= 0.
\end{aligned}$$

Define

$$\beta \equiv R(1+c). \tag{43}$$

Replacing β for $R(1+c)$ we can also write the cubic equation for the partial regulation price as follows:

$$2d(\sigma R-1-c)P^{*3}+2d\beta P^{*2}+R(\sigma qR-2d\beta)P^{*}-qR\beta=0 \tag{44}$$

It is easy to show that this cubic equation has only one real root and two complex conjugate roots. The only real root can easily be obtained using Vieta's substitution for cubic equations.

7.4 Proofs omitted in the main text

Lemma 1. *The fire sale price of risky asset, $P(n, b)$, is decreasing in n and increasing in b.*

Proof. The asset market clearing condition in the bad state at $t = 1$ is given as

$$Q^s(P) = \frac{c-b}{P}n = Q^d(P),$$

which can be written as

$$(c-b)n = PQ^d(P). \tag{45}$$

First, take the partial derivative of both sides of this last equation with respect to n:

$$\begin{aligned} c-b &= \frac{\partial P}{\partial n}Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\frac{\partial P}{\partial n}, \\ &= \frac{\partial P}{\partial n}\left\{Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\right\}, \\ &= \frac{\partial P}{\partial n}Q^d(P)\left\{1+\epsilon^d\right\}, \end{aligned}$$

where

$$\epsilon^d = \frac{\partial Q^d(P)}{\partial P}\frac{P}{Q^d},$$

is the price elasticity of outside investors' demand function. Rearranging the last equation gives

$$\frac{\partial P}{\partial n} = \frac{c-b}{Q^d(P)(1+\epsilon^d)} < 0$$

since $1 + \epsilon^d < 0$ by the *Elasticity* assumption, and $c - b > 0$ by assumption here because we are examining the case with fire sales. We later show in Proposition 1 and 4 that $c - b > 0$ actually holds in equilibrium.

For the second part of the proof take the partial derivative of both sides of (45) with respect to b:

$$\begin{aligned} -n &= \frac{\partial P}{\partial b}Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\frac{\partial P}{\partial b}, \\ &= \frac{\partial P}{\partial b}\left\{Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\right\}, \\ &= \frac{\partial P}{\partial b}Q^d(P)\left\{1+\epsilon^d\right\}. \end{aligned}$$

Rearranging the last equation gives

$$\frac{\partial P}{\partial b} = -\frac{n}{Q^d(P)(1+\epsilon^d)} > 0.$$

because $1 + \epsilon^d < 0$ by *Elasticity* assumption.

□

Lemma 2. *The fraction of risky assets sold, $1-\gamma(n,b)$, is increasing in n and decreasing in b.*

Proof. Using (3) we can write banks' asset sales in equilibrium as $1-\gamma(n,b) = (c-b)/P(n,b)$. Note that

$$\frac{\partial(1-\gamma)}{\partial n} = \frac{\partial(1-\gamma)}{\partial P}\frac{\partial P}{\partial n} > 0,$$

because $\partial(1-\gamma)/\partial P = -c/P^2 < 0$ from (3) and by Lemma 1 we have that $\partial P/\partial n < 0$. Similarly, we can obtain

$$\frac{\partial(1-\gamma)}{\partial b} = -\frac{1}{P} + \frac{\partial(1-\gamma)}{\partial P}\frac{\partial P}{\partial b} < 0,$$

since $\partial(1-\gamma)/\partial P < 0$ as shown above, and by Lemma 1 we have that $\partial P/\partial b > 0$. □

Proposition 1. *Banks take fire sale risk in equilibrium; that is, $b_i < c$ for all banks.*

Proof. It is straightforward to show that banks never carry excess liquidity in equilibrium, that is, $b_i > c$. This is because when $b_i > c$ the liquid assets in excess of the shock, $(b_i - c)n$, have no use even in the bad state; the expected return on liquid assets is equal to one and dominated by the expected return on the risky asset, $R - cq$, by the *Technology* assumption. Therefore, for contradiction assume that $b_i = c$. Corresponding first order conditions of bank's problem given by (6) with respect to n_i and b_i are respectively:

$$\begin{aligned}(1-q)(R+b_i) + qR &= D^{'}(n_i(1+b_i))(1+b_i),\\ (1-q)n_i + qn_i &= D^{'}(n_i(1+b_i))n_i.\end{aligned}$$

The last equation implies that $D^{'}(n_i(1+b_i)) = 1$. Substitute this into the first equation to obtain $R + (1-q)b_i = 1 + b_i$. Now using $b_i = c$ gives $R + (1-q)c = 1 + c$, which contradicts with the *Technology* assumption, that is, $R > 1 + cq$. Therefore, we must have $b_i < c$ for all $i \in [0,1]$. □

Proposition 2. *The competitive equilibrium price of assets is given by*

$$P = \frac{qR(1+c)}{R-1+q}.$$

The equilibrium price, P, is increasing in the probability of the liquidity shock, q, and the size of the shock, c, but decreasing in the return on the risky assets, R.

Proof. The first order conditions of the banks' problem (6) with respect to n_i and b_i respectively are:

$$(1-q)(R+b_i) + qR\gamma_i = D^{'}(n_i(1+b_i))(1+b_i), \tag{46}$$

$$(1-q)n_i + qR\frac{1}{P}n_i = D^{'}(n_i(1+b_i))n_i, \tag{47}$$

where $\gamma_i = 1 - (c - b_i)/P$ as obtained in the previous section. Combining the two equations gives:

$$(1-q)R + (1-q)b_i + qR + qR(\frac{b_i - c}{P}) = (1-q) + (1-q)b_i + \frac{qR}{P} + \frac{qR}{P}b_i.$$

In this last equation, the terms that involve the liquidity ratio, b_i, on both sides cancel out each other, and hence we can solve for P, the competitive equilibrium price of assets. It is straightforward to obtain the sign of the derivative of the equilibrium price with respect to model parameters, R, c, q. □

Proposition 3. *The comparative statics for the competitive equilibrium risky investment level, n, and liquidity ratio, b, are as follows:*

1. *The risky investment level (n), is increasing in the return on the risky asset (R), and decreasing in the size of the liquidity shock (c), probability of the bad state (q), and the marginal cost of funds (d).*

2. *The liquidity ratio (b), is increasing in the return on the risky asset (R), size of the liquidity shock (c), and the probability of the bad state (q), and decreasing in the marginal cost of funds (d).*

Proof. The derivatives below use the following closed-form solution for the competitive equilibrium risky investment level and liquidity ratio as obtained in Section 7.1:

$$n = \frac{[R-1-qc][R-1+q+2dR(1+c)]}{(R-1+q)(1+c)^2 2d}, \qquad b = \frac{cq - \frac{2dR}{\tau+1}}{q + \frac{2dR}{\tau+1}}.$$

In most derivatives below, we use the Technology assumption ($R-1-qc>0$) to obtain the sign. The derivatives for the risky investment level and their signs can be obtained as follows after some algebraic manipulation:

$$\begin{aligned}
\frac{\partial n}{\partial R} &= \frac{(R-1+q)^2 + 2d(1+c)[(R+q-1)^2 + (1-q)q(1+c)]}{(R-1+q)^2(1+c)^2 2d} > 0. \\
\frac{\partial n}{\partial c} &= \frac{-[2(R-1)+q(1-c)+2dR(1+c)]}{2d(1+c)^3} < 0. \\
\frac{\partial n}{\partial q} &= \frac{-c(R-1+q)^2 - 2dR(1+c)(R-1)(1+c)}{(R-1+q)^2(1+c)^2 2d} < 0. \\
\frac{\partial n}{\partial d} &= \frac{-(R-1-qc)(R-1+q)}{2(R-1+q)(1+c)^2 d^2} < 0.
\end{aligned}$$

Similarly, the derivatives for the liquidity ratio and their signs can be obtained as follows:

$$\frac{\partial b}{\partial R} = \frac{\frac{2d(1-q)}{(\tau+1)^2}}{[\frac{2dR}{\tau+1}+q]^2} > 0. \qquad \frac{\partial b}{\partial c} = \frac{q^2}{[\frac{2dR}{\tau+1}+q]^2} > 0.$$

$$\frac{\partial b}{\partial q} = \frac{\frac{2dR}{(\tau+1)^2}}{[\frac{2dR}{\tau+1}+q]^2} > 0. \qquad \frac{\partial b}{\partial d} = \frac{-\frac{2R}{\tau+1}q(1+c)}{[\frac{2dR}{\tau+1}+q]^2} < 0.$$

□

Proposition 4. *It is optimal for the constrained planner to take fire sale risk; that is, the constrained optimal liquidity ratio satisfies $b < c$.*

Proof. In principle, it is possible to completely insure against the fire sale risk. Under full insurance, similar to some interpretation of narrow banking (Freixas and Rochet, 2008, Chapter 7.2.2), the banks are able to cover the liquidity need even in the worst scenario by using their liquid holdings. However, we show that full insurance is not optimal and the constrained social planner takes some fire sale risk, by setting the aggregate liquidity ratio less than the liquidity need in the bad state, that is, by setting $b < c$.

To show this, we start with the full insurance case, that is $b = c$, and move ε amount of investment from liquid asset to risky asset, and show that this reallocation is socially profitable. Banks get exposed to fire sale risk as a result of this reallocation. First, we rewrite expected social welfare function in terms of the aggregate level of liquid assets, defined as $B \equiv bn$, rather than the liquidity ratio, b. We also ignore the expected utility of consumers and the expected profits of outsider investors for now for the sake of creating a benchmark. We incorporate those into the social welfare function below to complete the proof.

$$\begin{aligned} W &= (1-q)(R+b)n + qR\left(1 - \frac{c-b}{P}\right)n - D(n+nb), \\ &= (1-q)Rn + (1-q)B + qRn - qR\frac{cn}{P} + qR\frac{B}{P} - D(n+B), \\ &= Rn + (1-q)B - qR\frac{cn}{P} + qR\frac{B}{P} - D(n+B). \end{aligned}$$

In case of perfect insurance the size of liquidity hoarded at the initial period is equal to the size of the liquidity need in the bad state, that is, $B = cn$. Expected social welfare in the full insurance case boils down to $W_{fi} = Rn + (1-q)B - D(n+B)$. Now, moving some amount of funds in the initial period from liquid assets to the risky investment, yields $B_{new} = B - \varepsilon$ and $n_{new} = n + \varepsilon$. Let us denote the fire sale price after the reallocation by P_ε. Expected social welfare changes as follows after the reallocation of funds

$$\begin{aligned} W_{new} &= Rn_{new} + (1-q)B_{new} - qR\frac{cn_{new}}{P_\varepsilon} + qR\frac{B_{new}}{P_\varepsilon} - D(n_{new} + B_{new}), \\ &= R(n+\varepsilon) + (1-q)(cn-\varepsilon) - qR\frac{c(n+\varepsilon)}{P_\varepsilon} + qR\frac{cn-\varepsilon}{P_\varepsilon} - D(n+\varepsilon+B-\varepsilon), \\ &= Rn + (1-q)B - qR\frac{cn}{P_\varepsilon} + qR\frac{B}{P_\varepsilon} + R\varepsilon - (1-q)\varepsilon - qR\varepsilon\frac{1+c}{P_\varepsilon} - D(n+B), \\ &= Rn + (1-q)B - D(n+B) + \varepsilon\left(R - 1 + q - qR\frac{1+c}{P_\varepsilon}\right), \\ &= W_{fi} + \varepsilon\left(R - 1 + q - qR\frac{1+c}{P_\varepsilon}\right). \end{aligned} \tag{48}$$

Thus, $W_{new} > W_{fi}$ if and only if $P_\varepsilon > \frac{qR(1+c)}{R-1+q} \equiv \bar{P}_c$. In other words, as long as the fire sale price does not decrease dramatically as a result of a small amount of fire sales, taking some fire sale risk is socially optimal.

$R-1+q$ in equation (48) is the benefit of reallocating the funds from liquid asset to risky asset while $qR\frac{1+c}{P_\varepsilon}$ represents the expected cost of this reallocation. Though the benefit is constant, the cost is decreasing in P_ε. Therefore, lower fire sale price makes this reallocation of funds more costly. Thus, the reallocation is optimal as long as it is not very costly to do so, as long as the price does not decrease below a certain threshold which we solved as $\bar{P}_c$. This reallocation (i.e. deviating from the full insurance) is optimal in our assumptions on the outside investors because our assumptions yield a smooth demand curve with a P_ε close to R as we have $F'(0) = R$.

The social welfare function above does not include the profits of outside investors. Incorporating the profits of outside investors as well as the expected utility of consumers into the welfare function changes the constrained planner's problem as follows:

$$
\begin{aligned}
W &= (1-q)(R+b)n + qR\left(1 - \frac{c-b}{P}\right)n - D(n+nb) + q[F(y) - Py] + e, \\
&= (1-q)(R+b)n + qR\left(1 - \frac{c-b}{P}\right)n - D(n+nb) + q[F(y) - (cn - B)] + e,
\end{aligned}
$$

where we use the market clearing condition (29) to substitute $Py = (c-b)n = cn - B$. Similarly, we can show that the expected social welfare changes as follows after the reallocation of ε unit of funds from the risky to liquid assets:

$$
\begin{aligned}
W_{new} &= W_{fi} + \varepsilon(R - 1 + q) - qR\frac{(1+c)\varepsilon}{P_\varepsilon} + qF(y_\varepsilon) - q(1+c)\varepsilon, \\
&= W_{fi} + \varepsilon(R - 1 + q) - qR\frac{(1+c)\varepsilon}{P_\varepsilon} + q[F(y_\varepsilon) - y_\varepsilon P_\varepsilon],
\end{aligned}
$$

where we use $\frac{(1+c)\varepsilon}{P_\varepsilon} = y_\varepsilon$. Note that the social welfare under full insurance now is equal to $W_{fi} = Rn + (1-q)B - D(n+B) + e$ as the expected profit of outsiders is zero under full insurance. The following equation provides the indifference condition between deviating from full insurance or not, for the social planner.

$$
\varepsilon(R - 1 + q) - qR\frac{(1+c)\varepsilon}{P_\varepsilon} + q[F(y_\varepsilon) - y_\varepsilon P_\varepsilon] = 0.
$$

This equation yields the cutoff level for the fire sale price, as a result of a tiny deviation from full insurance, above which the deviation is socially profitable. Note that the benefit of reallocating funds now is larger compared to one depicted equation (48) as the additional term, $q[F(y_\varepsilon) - y_\varepsilon P_\varepsilon]$ the expected profit of outside investors, is positive. Thus, the cutoff level $\bar{P}$ here is lower compared to the one above: $\bar{P} < \bar{P}_c = \frac{qR(1+c)}{R-1+q}$.

Therefore, as long as price does not suddenly drop to $\bar{P}$ as a result of a marginal exposure to fire sales, taking fire sale risk is optimal for the constrained planner. □

Proposition 5. *Competitive equilibrium allocations compare to the constrained efficient allocations as follows:*

1. *Risky investment levels:* $n > n^{**}$
2. *Liquidity ratios:* $b < b^{**}$

Proof. We defer the proof of this proposition to Lemmas 4 and 5, which are under the proof of Proposition 7 below. □

Proposition 6. *Let the operational cost of a bank be given by* $\Phi(x) = dx^2$. *Then, for any technology for outside investors* F *that satisfies the Concavity, Elasticity, and Regularity assumptions with* $F'(0) = R$, *banks decrease their liquidity ratio as the regulator tightens capital requirements; that is,* $b'(n) \geq 0$.

Proof. We are studying the partial regulation case, in which banks are free to choose their liquidity ratio b_i but the regulator limits their choice of n_i. Therefore, we can write banks' expected profit function at $t = 0$ as follows

$$\max_{b_i} \Pi_i(b_i; n) = (1-q)\{R + b_i\}n + qR\gamma_i n - D(n(1+b_i)). \tag{49}$$

Here, we can treat n like a parameter of the model because banks take it as given. The regulator, in a sense, determines the aggregate amount of n. Therefore, the first order conditions of the banks' problem above is

$$\begin{aligned}\frac{\partial \Pi(b_i; n)}{\partial b_i} &= (1-q)n + qRn\frac{\partial \gamma}{\partial b_i} - n - 2dn^2(1+b_i) = 0 \\ &= (1-q)n + qRn\frac{1}{P} - n - 2dn^2(1+b_i) = 0,\end{aligned}$$

which can be simplified as

$$q\left(\frac{R}{P} - 1\right) = 2dn(1+b_i). \tag{50}$$

Note that we can obtain the derivative of the equilibrium price with respect to the regulatory parameter, n, as follows:

$$\frac{\partial P}{\partial n} = \frac{F_2(c - b_i)}{F_1 + yF_2}, \tag{51}$$

where $F_k \equiv \frac{d^k F(y)}{dy^k}$ for $k = 1, 2$, and y shows the quantity of assets sold to outside investors in fire sales.

Banks' profit function exhibits increasing differences in b_i and n if its cross derivative is positive. Increasing differences mean that $b'(n) > 0$, that is, the optimal choice of b_i in banks' problem is increasing with the regulatory parameter, n. We can obtain the cross derivative of banks' expected profit as

$$\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} = (1-q) + qR\left(\frac{1}{P} - \frac{n}{P^2}\frac{\partial P}{\partial n}\right) - 1 - 4dn(1+b_i).$$

Substituting for $dn(1+b_i)$ from the banks' first-order condition (50) and using the expression for $\partial P/\partial n$, given by (51), and we can simplify the cross derivative of banks' expected profit as follows

$$\begin{aligned}\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} &= (1-q) + qR\left(\frac{1}{P} - \frac{n}{P^2}\frac{F_2(c-b_i)}{F_1 + yF_2}\right) - 1 - 2q\left(\frac{R}{P} - 1\right), \\ &= -q + qR\left(\frac{1}{P} - \frac{n(c-b_i)}{Py}\frac{1}{P}\frac{yF_2}{F_1 + yF_2}\right) - \frac{2qR}{P} + 2q, \\ &= q + qR\left(\frac{1}{P} - \frac{n(c-b_i)}{Py}\frac{1}{P}\frac{yF_2}{F_1 + yF_2}\right) - \frac{2qR}{P},\end{aligned}$$

where in the second line we manipulated the second term within the parentheses by multiplying and dividing by y. Now, use of the equality of $y = n(c - b_i)/P$ in equilibrium and finally substitute

$P = F_1$ to get:

$$\begin{aligned}\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} &= q + qR\left(\frac{1}{P} - \frac{1}{P}\frac{yF_2}{F_1 + yF_2}\right) - \frac{2qR}{P} = q - qR\left(\frac{1}{P} + \frac{1}{P}\frac{yF_2}{F_1 + yF_2}\right) \\ &= q\left\{1 - R\frac{[F_1 + 2yF_2]}{F_1(F_1 + yF_2)}\right\}.\end{aligned}$$

Increasing differences hold if

$$\frac{\partial^2 \Pi(b_i; n)}{\partial b_i \partial n} > 0 \Leftrightarrow R < \frac{F_1(F_1 + yF_2)}{F_1 + 2yF_2} \equiv \kappa. \tag{52}$$

Therefore, if we assume that outside investors' technology F satisfies (52), we are done. If we do not make this assumption, we can instead assume that $F_1(0) = R$ and show that (52) holds for all $y > 0$. Note that when y is equal to zero κ is equal to F_1 by definition, and we have that $F_1(0) = R$ by assumption. Therefore, in order to show that $\kappa > R$ for all $y > 0$, all we need to show is that κ is increasing in y. Below we show that the derivative of κ with respect to y is indeed positive:

$$\begin{aligned}\frac{d\kappa}{dy} &= \frac{[F_2(F_1 + yF_2) + F_1(F_2 + F_2 + yF_3)][F_1 + 2yF_2]}{(F_1 + 2yF_2)^2}, \\ &= \frac{[3F_1F_2 + yF_2^2 + F_1F_3y][F_1 + yF_2 + yF_2] - [F_1(F_1 + yF_2)][F_2 + 2F_2 + 2yF_3]}{(F_1 + 2yF_2)^2}.\end{aligned} \tag{53}$$

Because the denominator of the derivative is positive we focus on the numerator to obtain the sign of the derivative. The numerator of (53) can be simplified as follows:

$$\begin{aligned}\frac{d\kappa}{dy} \times (F_1 + 2yF_2)^2 &= y(F_2^2 - F_1F_3)(F_1 + yF_2) + yF_2[3F_1F_2 + yF_2^2 + F_1F_3y], \\ &= y(F_2^2 - F_1F_3)F_1 + yF_2[yF_2^2 - yF_1F_3 + 3F_1F_2 + yF_2^2 + yF_1F_3], \\ &= y(F_2^2 - F_1F_3)F_1 + yF_2[3F_1F_2 + 2yF_2^2].\end{aligned}$$

Divide both sides with y to simplify further:

$$\begin{aligned}\frac{d\kappa}{dy} \times \frac{(F_1 + 2yF_2)^2}{y} &= F_1F_2^2 - F_1^2F_3 + 3F_1F_2^2 + 2yF_2^3 \\ &= 4F_1F_2 - F_1^2F_3 + 2yF_2^3 \\ &= 2F_1F_2 - F_1^2F_3 + 2F_1F_2 + 2yF_2^3 \\ &= F_1(2F_2^2 - F_1F_3) + 2F_2^2(F_1 + yF_2) > 0.\end{aligned}$$

$2F_2^2 - F_1F_3$ is positive due to the *Regularity* assumption, and $F_1 + yF_2$ is positive due to the *Elasticity* assumption. □

Proposition 7. *Risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium* (n, b, P, $1-\gamma$, $(1-\gamma)n$)*, partial regulation equilibrium* (n^*, b^*, P^*, $1-\gamma^*$, $(1-\gamma^*)n^*$)*, and complete regulation equilibrium* (n^{**}, b^{**}, P^{**}, $1-\gamma^{**}$, $(1-\gamma^{**})n^{**}$) *compare as follows:*

1. *Risky investment levels:* $n > n^{**} > n^*$
2. *Liquidity ratios:* $b^{**} > b > b^*$
3. *Financial stability measures*
   (a) *Price of assets in the bad state:* $P^{**} > P^* > P$
   (b) *Fraction of assets sold:* $1-\gamma > 1-\gamma^* > 1-\gamma^{**}$
   (c) *Total fire sales:* $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

Proof. Proof of this proposition is established through a series of lemmas below. □

Lemma 3. $P^{**} > P^* > P$

Proof. **Part 1:** $P^* > P$. First, note that we obtain the competitive equilibrium price of assets in the main text as:

$$P = \frac{qR(1+c)}{R-1+q} = \frac{\beta}{R\sigma},$$

using the definitions of σ, β from (39) and (43). Now, take the cubic equation given by (44) and divide it by $R\sigma$ to obtain:

$$R\{2d\sigma P^{*3} + (\sigma qR - 2d\beta)P^* - q\beta\} + 2d\beta P^{*2} - 2d(1+c)P^{*3} = 0$$
$$R\left[\frac{2d}{R}P^{*3} + \left(q - \frac{2d\beta}{\sigma R}\right)P^* - \frac{q\beta}{\sigma R}\right] + \frac{2d\beta}{\sigma R}P^{*2} - \frac{2d(1+c)}{\sigma R}P^{*3} = 0$$

Note that $(1+c)/\sigma = P$, and substitute this into the equation above and manipulate:

$$R\left[\frac{2d}{R}P^{*3} + (q - 2dP)\,P^* - qP\right] + 2dPP^{*2} - \frac{2d}{R}PP^{*3} = 0$$
$$R\left(\frac{2d}{R}P^{*2} + q\right)P^* - \left(2dRP^* + qR - 2dP^{*2} + \frac{2d}{R}P^{*3}\right)P = 0$$

From this last equivalence we can obtain the price ratios in these two cases as:

$$\frac{P}{P^*} = \frac{2dP^{*2} + qR}{\frac{2d}{R}P^{*3} - 2dP^{*2} + 2dRP^* + qR} = \frac{2dRP^{*2} + qR^2}{2dP^{*3} - 2dRP^{*2} + 2dR^2P^* + qR^2}, \tag{54}$$

In order to show that $P < P^*$, we need to show that the numerator of this ratio is less then its denominator, that is

$$\begin{aligned} 2dRP^{*2} + qR^2 &< 2dP^{*3} - 2dRP^{*2} + 2dR^2P^* + qR^2 \\ 4dRP^{*2} &< 2dP^*(P^{*2} + R^2) \\ 0 &< (R - P^*)^2 \end{aligned}$$

The last inequality holds because we must have $P^* < R$ in equilibrium. $P^* < R$ holds in equilibrium for the following reason: Assumption *Concavity* states that $P^* \leq R$, yet the equality cannot arise in equilibrium as $P^* = R$ implies $P = R$ as well due to (54). However $P < R$ holds due to the *Technology* assumption, $R - cq - 1 > 0$. Thus, we must have $P^* < R$.

Part 2: $P^{**} > P^*$. First, note that

$$R - 1 - qc = R - 1 + q - q - qc = R - 1 + q - q(1+c) = qR\sigma - q(1+c) = q(\sigma R - 1 - c),$$

where σ, β are defined by (39) and (43). Using this equivalence we can write the polynomial equation that gives P^{**}, equation (31), as

$$\begin{aligned}(R - 1 - qc)P^{**2} + q\beta P^{**} - qR\beta &= 0\\ q(\sigma R - 1 - c)P^{**2} + q\beta P^{**} - qR\beta &= 0\\ \frac{\sigma R - 1 - c}{R}P^{**2} + \frac{\beta}{R}P^{**} &= \beta\end{aligned}$$

Now substitute β using the last equation above into the cubic equation that gives P^*, equation (44):

$$\begin{aligned}2d(\sigma R - 1 - c)P^{*3} + 2d\beta P^{*2} + R(\sigma qR - 2d\beta)P^* - qR\beta &= 0\\ 2d(\sigma R - 1 - c)P^{*3} + 2d\beta P^{*2} + R\left(\sigma qR - 2d\frac{\sigma R - 1 - c}{R}P^{**2} - 2d\frac{\beta}{R}P^{**}\right)P^* - qR\beta &= 0\\ 2d(\sigma R - 1 - c)P^{*3} + 2d\beta P^{*2} + \sigma qR^2 P^* - 2d(\sigma R - 1 - c)P^{**2}P^* - 2d\beta P^{**}P^* - qR\beta &= 0\\ (2d\sigma R - 1 - c)P^*(P^{*2} - P^{**2}) + 2d\beta P^*(P^* - P^{**}) + qR(\sigma RP^* - \beta) &= 0 \quad (55)\end{aligned}$$

Note that the first two terms in (55) must have the same sign, which will be equal to the inverse of the sign of the last term, $qR(\sigma RP^* - \beta)$. Therefore, in order to show that $P^* - P^{**} < 0$, we need to show that $qR(\sigma RP^* - \beta) > 0$. We can write this last terms as

$$qR(\sigma RP^* - \beta) = qR^2\sigma P^* - q(1+c)R^2 > 0 \Leftrightarrow \sigma P^* - 1 - c > 0.$$

Note that by Part 1, we know that $P < P^*$. Hence, if $\sigma P - 1 - c \geq 0$ then we must necessarily have $\sigma P^* - 1 - c > 0$. Using the closed-form solution of the competitive equilibrium, given by (9), we can show that:

$$\sigma P - 1 - c = \frac{R - 1 + q}{qR}\frac{qR(1+c)}{R - 1 + q} - 1 - c = 0$$

Therefore, we must have $\sigma P^* - 1 - c > 0$, which implies that $P^{**} > P^*$ in order for equation (55) to hold. □

Lemma 4. $b^{**} > b > b^*$

Proof. **Part 1:** $b^{**} > b$. Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (25) and (32) as:

$$b = \frac{cq(\tau + 1) - 2dR}{q(\tau + 1) + 2dR}, \qquad b^{**} = \frac{cq(\tau^{**} + 1)^2 - 2dR}{q(\tau^{**} + 1)^2 + 2dR}.$$

Comparing the liquidity ratios under competitive equilibrium (b) and under the constrained planner's solution (b^{**}), we see that they have the same following functional form:

$$f(x) = \frac{cqx - 2dR}{qx + 2dR}. \tag{56}$$

The only difference is $x = \tau + 1$ in the competitive case versus $x = (\tau^{**} + 1)^2$ in the constrained planner's problem. First, note that

$$f'(x) = \frac{cq(qx + 2dR) - (cqx - 2dR)q}{(qx + 2dR)^2} = \frac{2dRq(1 + c)}{(qx + 2dR)^2} > 0. \tag{57}$$

Therefore, in order to show that $b^{**} > b$, all we need to show is that $(\tau^{**} + 1)^2 > \tau + 1$, which can be written equivalently as:

$$\frac{R^2}{P^{**2}} > \frac{R}{P} \Leftrightarrow P^{**2} < RP.$$

Now, substitute P^{**2} from the solution to the constrained planner's problem, given by (31) and the competitive equilibrium price, P, from (9) to write this inequality as:

$$\begin{aligned}
\frac{q\beta(R - P^{**})}{R - 1 - qc} &< R\frac{qR(1 + c)}{R - 1 + q} \\
R - P^{**} &< R\frac{R - 1 - qc}{R - 1 + q} \\
R\left(1 - \frac{R - 1 - qc}{R - 1 + q}\right) &< P^{**} \\
\frac{Rq(1 + c)}{R - 1 + q} = P &< P^{**}.
\end{aligned}$$

The last inequality holds by Lemma 3. Therefore, $(\tau^{**} + 1)^2 > \tau + 1$, which implies that $b^{**} > b$.

Part 2: $b > b^*$. Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (25) and (35)as:

$$b = \frac{cq(\tau + 1) - 2dR}{q(\tau + 1) + 2dR}, \qquad b^{**} = \frac{cq(\tau^* + 1) - 2dR}{q(\tau^* + 1) + 2dR}.$$

Comparing the liquidity ratios under competitive equilibrium (b) and under the partial regulation case (b^*), we see that they have the same functional form, $f(x)$, given above by (56). The only difference is $x = \tau + 1$ in the competitive case versus $x = \tau^* + 1$ in the partial case. We have shown above, by (57), that $f'(x) > 0$. Therefore, in order to show that $b > b^*$, all we need to show is that $\tau > \tau^*$. Note that because $\tau^* = R/P^* - 1$ and $\tau = R/P - 1$, and $P^* > P$ by Lemma 3, we have that $\tau > \tau^*$. This completes the proof. □

Lemma 5. $n > n^{**} > n^*$

Proof. **Part 1:** $n > n^{**}$. Using the closed-form solution for the competitive equilibrium, (26), and for the constrained planner's problem, (33), the difference in risky investment levels across these two cases can be written as

$$\begin{aligned}
n - n^{**} &= \frac{\tau}{\tau+1}\frac{q(\tau+1)+2dR}{2d(1+c)} - \frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2+2dR}{2d(1+c)} \\
&= \frac{1}{2d(1+c)}\left\{\frac{\tau[q(\tau+1)+2dR]}{(\tau+1)} - \frac{\tau^{**}[q(\tau^{**}+1)^2+2dR]}{(\tau^{**}+1)}\right\} \\
&= \frac{1}{2d(1+c)}\left\{q\tau + 2dR\frac{\tau}{(\tau+1)} - q\tau^{**}(\tau^{**}+1) - 2dR\frac{\tau^{**}}{(\tau^{**}+1)}\right\} \\
&= \frac{1}{2d(1+c)}\left\{q\tau - q\tau^{**}(\tau^{**}+1) + 2dR\frac{\tau}{(\tau+1)} - 2dR\frac{\tau^{**}}{(\tau^{**}+1)}\right\}
\end{aligned}$$

First, note that $\tau = R/P - 1 > \tau^{**} = R/P^{**} - 1$ by Lemma 3, and this implies that $2dR\frac{\tau}{(\tau+1)} - 2dR\frac{\tau^{**}}{(\tau^{**}+1)}$ is positive. Therefore, $n - n^{**}$ is positive if $q\tau - q\tau^{**}(\tau^{**}+1) \geq 0$. Next, we show that this inequality indeed holds. From (30) we have $R - 1 - qc = \frac{qR(R-P^{**})(1+c)}{P^2}$, which implies that:

$$\tau = \frac{R-1-qc}{q(1+c)} = \frac{R(R-P^{**})}{P^{**2}} = \frac{R}{P^{**}}(\frac{R}{P^{**}} - 1) = \tau^{**}(\tau^{**}+1),$$

where we use that $\tau = R/P - 1$ and $P = \frac{qR(1+c)}{R-1+q}$, as given by 9.

Part 2: $n^{**} > n^*$. For the second part of this lemma, we use the fact that $P^{**} > P^*$ as shown by Lemma 3. Using the closed-form solution for n^{**} from (33) and n^* from (36), we can write the difference in risky investment levels across these two cases as:

$$\begin{aligned}
n^{**} - n^* &= \frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2+2dR}{2d(1+c)} - \frac{\tau^*}{\tau^*+1}\frac{q(\tau^*+1)+2dR}{2d(1+c)}, \\
&= \frac{1}{2d(1+c)}\left\{\frac{\tau^{**}}{\tau^{**}+1}[q(\tau^{**}+1)^2+2dR] - \frac{\tau^*}{\tau^*+1}[q(\tau^*+1)+2dR]\right\}, \\
&= \frac{\Theta}{2d(1+c)(\tau^*+1)(\tau^{**}+1)} \qquad (58)
\end{aligned}$$

where

$$\begin{aligned}
\Theta &\equiv q(\tau^{**}+1)(\tau^*+1)[\tau^{**}(\tau^{**}+1) - \tau^*] + 2dR[\tau^{**}(\tau^*+1) - \tau^*(\tau^{**}+1)] \\
&= q(\tau^{**}+1)(\tau^*+1)[\tau - \tau^*] + 2dR[\tau^{**} - \tau^*], \qquad (59)
\end{aligned}$$

where we use the equivalence, $\tau = \tau^{**}(\tau^{**}+1)$, obtained in Part 1 above. Since the denominator of (58) is positive, in order to prove that $n^{**} - n^* > 0$, it suffices to show that $\Theta > 0$. In order to show that this inequality holds, first, we would like to write $2dR$ in θ in terms of τ's. For that start from the cubic equation that gives the partial equilibrium price as obtained by (44):

$$
\begin{aligned}
0 &= \frac{2d}{q}(R-1-qc)P^{*3} + 2dR(1+c)P^{*2} + R^2(\sigma q - 2d(1+c))P^* - (1+c)qR^2, \\
0 &= \frac{2d}{q}q(1+c)\tau P^{*3} + 2dR(1+c)P^{*2} + R(R-1+q-2dR(1+c))P^* - (1+c)qR^2, \\
0 &= 2d(1+c)\tau P^{*3} + 2dR(1+c)P^{*2} + R\left(\frac{qR(1+c)}{P} - 2dR(1+c)\right)P^* - (1+c)qR^2, \\
0 &= 2d(1+c)\tau P^{*3} + 2dR(1+c)P^{*2} + \frac{qR^2(1+c)}{P}P^* - 2dR^2(1+c)P^* - (1+c)qR^2, \\
0 &= 2d(1+c)\tau\frac{R^3}{(\tau^*+1)^3} + 2dR(1+c)\frac{R^2}{(\tau^*+1)^2} + \frac{qR^2(1+c)}{P}\frac{R}{\tau^*+1} - 2dR^2(1+c)\frac{R}{\tau^*+1} - (1+c)qR^2, \\
0 &= 2d(1+c)\tau\frac{R^3}{(\tau^*+1)^3} + 2d(1+c)\frac{R^3}{(\tau^*+1)^2} + qR^2(1+c)\frac{\tau+1}{R}\frac{R}{\tau^*+1} - 2d(1+c)\frac{R^3}{\tau^*+1} - (1+c)qR^2, \\
0 &= \frac{2d(1+c)R^3}{(\tau^*+1)^3}[\tau+\tau^*+1-(\tau^*+1)^2] + qR^2(1+c)\left[\frac{\tau+1}{\tau^*+1} - 1\right], \\
0 &= \frac{2dR}{(\tau^*+1)^2}[\tau - \tau^*(\tau^*+1)] - q(\tau^*-\tau)
\end{aligned}
$$

where in the first line we use definition of σ, given by (39), to write $\sigma R - 1 - c = (R-1-qc)/q$, while using $\tau = R/P - 1 = (R-1-qc)/[q(1+c)]$ in the second line. In the third line we replaced $R - 1 + q$ with $\frac{qR(1+c)}{P}$ using equation (9) for price in competitive equilibrium and later we use $P^* = R/(\tau^*+1)$ to replace P^*. From the last equation above we can obtain:

$$
2dR = \frac{q(\tau^*+1)^2(\tau^*-\tau)}{\tau - \tau^*(\tau^*+1)} = \frac{q(\tau^*+1)^2(\tau^*-\tau)}{\tau^{**}(\tau^{**}+1) - \tau^*(\tau^*+1)},
$$

where we use the equivalence, $\tau = \tau^{**}(\tau^{**}+1)$, again. Now we plug this expression for $2dR$ back into (59) and show below that $\Theta > 0$ holds:

$$
\begin{aligned}
q(\tau^{**}+1)(\tau^*+1)[\tau-\tau^*] &> 2dR[\tau^*-\tau^{**}] = \frac{q(\tau^*+1)^2(\tau^*-\tau)}{\tau^{**}(\tau^{**}+1)-\tau^*(\tau^*+1)}[\tau^*-\tau^{**}] \\
\tau^{**}+1 &> \frac{(\tau^*+1)(-1)(\tau^*-\tau^{**})}{\tau^{**}(\tau^{**}+1)-\tau^*(\tau^*+1)} \\
\tau^{**}+1 &> \frac{(\tau^*+1)(\tau^*-\tau^{**})}{\tau^*(\tau^*+1)-\tau^{**}(\tau^{**}+1)} \\
(\tau^{**}+1)\tau^*(\tau^*+1) - \tau^{**}(\tau^{**}+1)^2 &> (\tau^*+1)(\tau^*-\tau^{**}) \\
(\tau^*+1)[\tau^*(\tau^{**}+1) - (\tau^*-\tau^{**})] &> \tau^{**}(\tau^{**}+1)^2 \\
(\tau^*+1)\tau^{**}(\tau^*+1) &> \tau^{**}(\tau^{**}+1)^2 \\
(\tau^*+1)^2 &> (\tau^{**}+1)^2.
\end{aligned}
$$

This inequality is true because $P^{**} > P^*$, as shown by Lemma 3, which implies that $\tau^* > \tau^{**}$, using the definitions $\tau^* = R/P^* - 1$ and $\tau^{**} = R/P^{**} - 1$. □

Lemma 6. $1 - \gamma > 1 - \gamma^* > 1 - \gamma^{**}$

Proof.

$$1-\gamma = \frac{c-b}{P} \text{ together with } b^{**} > b^* \text{ and } P^{**} > P^* \implies 1-\gamma^* > 1-\gamma^{**}$$

To obtain $(1-\gamma) > (1-\gamma^*)$, we can equivalently show that $\frac{\frac{c-b}{P}}{\frac{c-b^*}{P^*}} > 1$.

Using equations (25) and (35) for b and b^* respectively, $b = \frac{qc(\tau+1)-2dR}{2dR+q(\tau+1)} \implies c-b = \frac{2dR(1+c)}{2dR+q(\tau+1)}$, and similarly we can derive $c-b^* = \frac{2dR(1+c)}{2dR+q(\tau^*+1)}$. Writing τ and τ^* in terms of P and P^* we get the following,

$$\frac{\frac{c-b}{P}}{\frac{c-b^*}{P^*}} = \frac{c-b}{c-b^*}\frac{P}{P^*} = \frac{2dP^*+q}{2dP+q}\frac{P}{P^*}\frac{P^*}{P} > 1.$$

The last inequality holds because $P^* > P$ by Lemma 3. □

Lemma 7. $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

Proof. Given that the demand function for risky assets in the interim period is downward sloping (continuous and differentiable as well), the prices disclose the amount of fire sales. Hence, we can use the results in Lemma 3 to prove this lemma:

$$(1-\gamma)n = \frac{R}{P} - 1 \text{ and } P^{**} > P^* > P \implies (1-\gamma^{**})n^{**} < (1-\gamma^*)n^* < (1-\gamma)n.$$

□

Proposition 8. *Bank balance sheet sizes across different regimes are as follows:*

$$n(1+b) = n^{**}(1+b^{**}) > n^*(1+b^*)$$

Proof. Using the closed-form solutions in Sections 7.1 and 7.2, we can write the bank size under the competitive equilibrium and constrained planner's problem as follows:

$$n(1+b) = \frac{\tau}{\tau+1}\frac{2dR+q(\tau+1)}{2d(1+c)}\frac{q(\tau+1)(1+c)}{2dR+q(\tau+1)} = \frac{\tau}{\tau+1}\frac{q(\tau+1)}{2d} = \frac{q\tau}{2d}.$$

$$n^{**}(1+b^{**}) = \frac{\tau^{**}}{\tau^{**}+1}\frac{2dR+q(\tau^{**}+1)^2}{2d(1+c)}\frac{q(\tau^{**}+1)^2(1+c)}{2dR+q(\tau^{**}+1)^2} = \frac{q\tau^{**}(\tau^{**}+1)}{2d}.$$

Above we use equations (26) and (25) for the balance sheet size in competitive equilibrium and equations (32) and (33) for the constrained planner's case. Note that in Part I of Lemma 5 we show that $\tau = \tau^{**}(\tau^{**}+1)$. Thus, comparing the equations above we conclude $n(1+b) = n^{**}(1+b^{**})$.

Lastly, $b^{**} > b > b^*$, as shown in Lemma 4, and $n > n^{**} > n^*$, as shown in Lemma 5, together imply that $n(1+b) > n^*(1+b^*)$, that is, the bank balance sheet size is the smallest under partial regulation. □

Proposition 9. *Banks do not choose the constrained optimal risky investment level, n^{**}, if the regulator sets the minimum liquidity ratio at the constrained optimal level, b^{**}, that is, $n_i(b^{**}) \neq n^{**}$.*

Proof. For this proof we compare the first-order condition of the constrained planner's problem with respect to n, given by (11), and the first-order condition of banks' problem with respect to n_i, given by (17) when only liquidity is regulated. We reproduce these two first-order conditions below for convenience:

$$\begin{aligned}\Psi &\equiv (1-q)(R+b)+qR\Big\{\gamma+\frac{\partial\gamma}{\partial n}n\Big\}+q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}-D^{'}(\cdot)(1+b)=0,\\ \Upsilon &\equiv (1-q)(R+b)+qR\gamma_i-D^{'}(\cdot)(1+b)=0.\end{aligned}$$

The constrained planner's first-order condition, Ψ, includes extra terms because planner internalizes the effect of portfolio choices on asset prices and incorporated the well being of outside investors. These extra terms are:

$$Z=qR\frac{\partial\gamma}{\partial n}n+q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}$$

Hence, we can write $\Psi=\Upsilon+Z$. We first show that the sum of these extra terms is negative:

$$\begin{aligned}Z &= qR\frac{\partial\gamma}{\partial n}n+q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}\\ &= qR\frac{\partial\gamma}{\partial n}n+q\left\{P\left(\frac{c-b}{P}-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}\\ &= qR\frac{\partial\gamma}{\partial n}n+q\left\{c-b-\frac{\partial\gamma}{\partial n}nP-c+b\right\}\\ &= qR\frac{\partial\gamma}{\partial n}n-qP\frac{\partial\gamma}{\partial n}=q\frac{\partial\gamma}{\partial n}n(R-P)<0,\end{aligned}$$

where we use that in equilibrium $F'((1-\gamma)n^{**})=P^{**}$. The sign of Z is negative because $R>P^{**}$ by the *Concavity* assumption, and $\partial\gamma/\partial n<0$ by Lemma 2.

$Z<0$ implies that banks' first-order condition, Υ, evaluated at the constrained efficient allocations, n^{**},b^{**} is positive, that is $\Upsilon(n^{**},b^{**})>0$. On the contrary, we have $\Upsilon(n(b^{**}),b^{**})=0$ by definition of optimality. Furthermore, we can show that Υ is decreasing in n for a given b, that is:

$$\frac{\partial\Upsilon}{\partial n}=qR\frac{\partial\gamma}{\partial n}-D^{''}(\cdot)(1+b)^2<0,$$

because $D^{''}(\cdot)>0$ by assumption and $\partial\gamma/\partial n<0$ by Lemma 2. Therefore, we must have $n(b^{**})>n^{**}$. □

8 Online Appendix: Closed-form solutions without investors

8.1 A closed-form solution for the constrained planner's problem

Proposition 4 allows us to focus on the case $b < c$ when analyzing the constrained planner's problem. The planner chooses $n, b \geq 0$ to solve:

$$\max_{n,b} W(n,b) = (1-q)\{R+b\}n + qR\gamma n - D(n(1+b)), \tag{60}$$

subject to the society's budget constraint at $t = 0$, $0 \leq (1+b)n \leq e+E$. The first order conditions of the planner's problem with respect to n and b are respectively:

$$\begin{aligned}(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial\gamma}{\partial n}n\Big\} &= D^{'}(n(1+b))(1+b), &(61)\\ (1-q)n + qR\frac{\partial\gamma}{\partial b}n &= D^{'}(n(1+b))n, &(62)\end{aligned}$$

where $\gamma = 1 + \frac{b-c}{P}$ from banks' problem in the bad state, as obtained in Section 3.1.2. Combine the two equations to obtain:

$$(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial\gamma}{\partial n}n\Big\} = \left[(1-q) + qR\frac{\partial\gamma}{\partial b}\right](1+b) = D^{'}(n(1+b))(1+b). \tag{63}$$

First, note that using the functional form for outside investors' demand, given by (22), in the market clearing condition (5) yields the price of assets in the bad state as a function of initial portfolio allocations:

$$E(P,n,b) = Q^d(P) - Q^s(P,n,b) = 0 \implies \frac{R-P}{P} = \frac{c-b}{P}n \implies P = R-(c-b)n. \tag{64}$$

Substituting $\frac{\partial\gamma}{\partial n} = -\frac{(b-c)^2}{P^2}$ and $\frac{\partial\gamma}{\partial b} = \frac{R}{P^2}$, and later $P = R-(c-b)n$ into (63) and simplifying yields:

$$\begin{aligned}\frac{(1-q)(R-1)}{qR} &= -1 + \frac{(c-b)P - (c-b)^2 n + R(1+b)}{P^2}, &(65)\\ \frac{(1-q)(R-1)}{qR} + 1 &= \frac{(c-b)[R-(c-b)n] + (c-b)^2 n + R(1+b)}{P^2},\\ \frac{R-1+q}{qR} &= \frac{R(c-b+1+b)}{P^2}.\end{aligned}$$

From this last equation we can solve for the price of assets under constrained planner's solution, P^{**}:

$$P^2 = \frac{q(c+1)R^2}{R-1+q} \implies P^{**} = R\sqrt{\frac{q(c+1)}{R-1+q}}$$

Note that there is a simple relationship between the competitive equilibrium price, given by (9), and the price under constrained planner's solution, captured by $P = P^{**2}/R$. We can define $\tau^{**} \equiv R/P^{**} - 1$ similar to (23) to represent the total amount of assets sold under fire sales to outside investors in terms of the model parameters, and write risky investment as a function of the liquidity ratio as $n^{**} = P^{**}\tau^{**}/(c-b)$ using the market clearing condition, similar to (64).

We use these equations to solve for the constrained efficient portfolio allocations n^{**}, b^{**}. For that start from the first order condition with respect to b given above by (62) :

$$\begin{aligned}
(1-q) + qR\frac{\partial\gamma}{\partial b} &= D^{'}(n(1+b)), \\
1-q+q\left(\frac{R}{P}\right)^2 &= 1+2dn(1+b), \\
1-q+q(\tau^{**}+1)^2 &= 1+2dn(1+b), \\
q\{(\tau^{**}+1)^2-1\} &= 2dn(1+b), \\
q\{(\tau^{**}+1+1)(\tau^{**}+1-1\} &= 2d\frac{P\tau^{**}}{c-b}(1+b), \\
q\tau^{**}(\tau^{**}+2) &= 2d\frac{R}{\tau^{**}+1}\frac{\tau^{**}}{c-b}(1+b), \\
q(\tau^{**}+1)(\tau^{**}+2)(c-b) &= 2dR(1+b), \qquad (66) \\
q(\tau^{**}+1)(\tau^{**}+2)c-2dR &= b\{2dR+q(\tau^{**}+1)(\tau^{**}+2)\}, \qquad (67)
\end{aligned}$$

where we use $R/P^{**} = \tau^{**}+1$ and $n^{**} = P^{**}\tau^{**}/(c-b)$. For future reference, using (66), we can obtain the liquidity shortage per risky asset in the constrained planner's solution as

$$c - b^{**} = \frac{2dR(1+b^{**})}{q(\tau^{**}+1)(\tau^{**}+2)}. \qquad (68)$$

We can obtain the closed-form solution for the constrained efficient liquidity ratio, b^{**}, by rearranging (67), as

$$b^{**} = \frac{cq(\tau^{**}+1)(\tau^{**}+2)-2dR}{2dR+q(\tau^{**}+1)(\tau^{**}+2)}. \qquad (69)$$

Finally, we can obtain the closed-form solution for the risky investment level by substituting b^{**} into $n^{**} = P^{**}\tau^{**}/(c-b)$ and using $P^{**} = R/(\tau^{**}+1)$

$$n^{**} = \frac{\tau^{**}}{\tau^{**}+1}\frac{2dR+q(\tau^{**}+1)(\tau^{**}+2)}{2d(1+c)}. \qquad (70)$$

8.2 A closed-form solution for the partial regulation case

In the partial regulation case, we consider the problem of a regulator who chooses the optimal level of risky investment, $n \geq 0$, at $t = 0$ to maximize the net expected social welfare but who allows banks to freely choose their liquidity ratio, b_i. The bank chooses the liquidity ratio, b_i, to maximize its expected profits; hence, the problem of the bank is as follows:

$$\max_{b_i} \Pi_i(b_i; n) = (1-q)\{R+b_i\}n + qR\gamma_i n - D(n(1+b_i). \qquad (71)$$

The first-order condition of the banks' problem (71) with respect to b_i is

$$1-q+qR\frac{1}{P} = D^{'}(n(1+b_i)). \qquad (72)$$

We use the same functional form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section 7.1 and constrained planner's problem in Section 8.1. We can also define $\tau^* \equiv R/P^* - 1$ similar to (23) to represent the total amount of assets sold under fire sales to outside investors in terms of the model parameters, and write risky investment as a function of the liquidity ratio as $n^* = P^*\tau^*/(c-b)$ using the market clearing condition, similar to (64) Now, use the functional-form for the operational cost in banks' first-order condition and manipulate

$$\begin{aligned} 1 - q + \frac{qR}{P} &= 1 + 2dn(1+b), \\ q\left(\frac{R}{P} - 1\right) &= 2d\frac{P\tau}{c-b}(1+b), \\ q\tau &= 2d\frac{R}{\tau+1}\frac{\tau}{c-b}(1+b), \end{aligned}$$

where we first use $n = \frac{P\tau}{c-b}$ and then substitute $P = \frac{R}{\tau+1}$. From the last equation we can obtain an expression for the liquidity ratio in this case in terms of τ^* as follows

$$b^* = \frac{qc(\tau^*+1) - 2dR}{q(\tau^*+1) + 2dR}. \tag{73}$$

Using $n = \frac{P\tau}{c-b}$ and $P = \frac{R}{\tau+1}$ once more, we can obtain a similar expression for the risky investment level in this case in terms of τ^* as follows:

$$n^* = \frac{\tau^*}{\tau^*+1}\frac{q(\tau^*+1) + 2dR}{2d(1+c)}. \tag{74}$$

All that remains now is to obtain a closed-form solution for $\tau^* = R/P^* - 1$, and substitute that in (73) and (74) to obtain closed-form solutions for n^* and b^*. To obtain a closed-form solution for P^* we analyze the regulator's problem. The regulator takes into account that for any given n, the banks optimally choose their liquidity ratio $b(n)$, as shown by the response function (14). Hence, we can write the regulator's objective function as:

$$\max_n \ W(n) = \ (1-q)\{R + b(n)\}n + qR\gamma n - D((1+b(n))n),$$

from which we can obtain the following first order conditions with respect to n as

$$(1-q)\{R + b(n) + nb'(n)\} + qR\left\{\gamma + n\frac{d\gamma}{dn}\right\} = D'(n(1+b))\{1 + b(n) + nb'(n)\}. \tag{75}$$

We use the same functional-form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section 7.1 and constrained planner's problem in Section 8.1. First, note that substituting for P using (64) into γ, given by (3), we get

$$\gamma = 1 + \frac{b(n) - c}{P} = 1 + \frac{b(n) - c}{R + (b(n) - c)n},$$

Using this equivalence, we can obtain the total derivative of γ with respect to n as:

$$\begin{aligned}\frac{d\gamma}{dn} &= \frac{\partial\gamma}{\partial b}b'(n) + \frac{\partial\gamma}{\partial n} \\ &= \frac{P-(b(n)-c)n}{P^2}b'(n) - \frac{(b(n)-c)^2}{P^2} \\ &= \frac{b'(n)}{P} - \frac{nb'(n)(b(n)-c)}{P^2} - \frac{(b(n)-c)^2}{P^2}. \end{aligned} \tag{76}$$

Replacing $d\gamma/dn$ in the first-order condition (75) with (76) and rearranging yields

$$(1-q)\{R+b(n)\} + qR\left(1+\frac{b(n)-c}{P}\right) + nb^{'}(n)\left\{1-q+\frac{qR}{P} - D'(\cdot) - \frac{(b(n)-c)n}{P^2}qR\right\}$$
$$-qR\frac{n(b(n)-c)^2}{P^2} - D^{'}(\cdot)\{1+b(n)\} = 0$$

We have that $1-q+qR/P-D'(\cdot)=0$ from the banks' first-order condition (72). Hence, the equation above can further be simplified as follows

$$\begin{aligned} R - qR + (1-q)b(n) + qR + qR\frac{b(n)-c}{P} - qR\frac{n(b(n)-c)^2}{P^2} - b^{'}(n)\{\frac{(b(n)-c)n^2}{P^2}qR\} & \\ -\left(1-q+\frac{qR}{P}\right)\{1+b(n)\} &= 0 \\ R-1+q+\frac{qR}{P}[b(n)-c-1-b(n)] - qR\frac{n(b(n)-c)^2}{P^2} - b^{'}(n)\{\frac{(b(n)-c)n^2}{P^2}qR\} &= 0 \\ R-1+q-\frac{qR(1+c)}{P} - qR\frac{n(b(n)-c)^2}{P^2} - b^{'}(n)\{\frac{(b(n)-c)n^2}{P^2}qR\} &= 0 \\ R-1+q-\frac{qR(1+c)[R+(b(n)-c)n] - qR(b(n)-c)^2n}{P^2} - b^{'}(n)\{\frac{(b(n)-c)n^2}{P^2}qR\} &= 0 \\ R-1+q-\frac{qR^2(1+c)}{P^2} - qR\frac{n(b(n)-c)(1+b(n))}{P^2} - b^{'}(n)\{\frac{(b(n)-c)n^2}{P^2}qR\} &= 0. \end{aligned} \tag{77}$$

Divide the last equation by qR to obtain

$$\frac{R-1+q}{qR} - \frac{R(1+c)}{P^2} - \frac{n(b(n)-c)(1+b(n))}{P^2} - b^{'}(n)\{\frac{(b(n)-c)n^2}{P^2}\} = 0. \tag{78}$$

Using $\sigma \equiv (R-1+q)/(qR)$ from 39, we can write the first-order condition of the regulator's problem (78) as

$$\frac{1}{P^2}\left\{\sigma P^2 - R(1+c) - n(b(n)-c)(1+b(n)) - b^{'}(n)(b(n)-c)n^2\right\} = 0. \tag{79}$$

We focus on the term inside the braces because in equilibrium price must be strictly positive. Using this term, we would like to write endogenous variables n and b in terms of the parameters of the model and P, and then, use these expression in the first-order conditions of the banks' problem (72) to obtain a closed-form solution for P. For that, first, below we obtain $1+b(n)$, $n(b(n)-c)$ and $b'(n)$ in terms of the parameters of the model and P starting from the banks' first-order condition

(72):

$$\begin{aligned}(1-q)+q\frac{R}{P} &= 1+2dn(1+b), \\ q(R-P) &= P2dn(1+b), \\ q(R-P) &= [R+(b-c)n]2dn(1+b), \\ -q(b-c)n &= 2dn(1+b)R+2dn(1+b)(b-c)n, \\ -(b-c)[q+2dn(1+b)] &= 2d(1+b)R,\end{aligned} \tag{80}$$

where we substitute for $P = R + (b-c)n$ using (64). Now, take the derivative of both sides with respect to n, and collect terms that involve $b^{'}(n)$:

$$\begin{aligned}-b'(n)[q+2dn(1+b)]-2d(b-c)[1+b+nb'(n)] &= 2dRb'(n), \\ -b'(n)[q+2dn(1+b)]-2d(b-c)(1+b)-2d(b-c)nb'(n) &= 2dRb'(n), \\ -b'(n)[2dR+2dn(b-c)+q+2dn(1+b)] &= 2d(b-c)(1+b), \\ -b'(n)[2dR+q+2dn(2b+1-c)] &= 2d(b-c)(1+b).\end{aligned}$$

From the last equation we obtain:

$$b'(n) = \frac{-2d(b-c)(1+b)}{2dR+q+2dn(2b+1-c)}. \tag{81}$$

We further simplify $b'(n)$ in order to eliminate b from this expression. In order to do this simplification, note that first, from the market clearing condition at $t=1$, $P = R+(b-c)n$, as derived in (64), we can obtain that

$$b-c = -\frac{R-P}{n}. \tag{82}$$

Second, from the banks' first-order condition, given by (80), we can obtain that

$$1+b = \frac{q}{2dn}\left(\frac{R}{P}-1\right). \tag{83}$$

Use these values for $1+b$ and $b-c$ into (81) to write $b'(n)$ as a function of n, P and the parameters of the model as follows

$$\begin{aligned}b'(n) &= \frac{-2d(-1)\frac{R-P}{n}\frac{q}{2dn}\left(\frac{R}{P}-1\right)}{2dR+q-2d(R-P)+2d\frac{q}{2d}\left(\frac{R}{P}-1\right)}, \\ &= \frac{\frac{q}{n^2P}(R-P)^2}{\frac{1}{P}[2dRP+qP-2dP(R-P)+q(R-P)]}, \\ &= \frac{q(R-P)^2}{n^2[2dRP+qP-2dRP+2dP^2+qR-qP]}, \\ &= \frac{q(R-P)^2}{n^2[2dP^2+qR]}.\end{aligned} \tag{84}$$

Eventually, use the expressions obtained for $1+b(n)$, $n(b(n)-c)$ and $b'(n)$ above to rewrite the term inside the braces in (79) as:

$$\sigma P^2 - R(1+c) + (R-P)\frac{q(R-P)}{2dPn} + \frac{q(R-P)^2}{n^2[2dP^2+qR]}\frac{R-P}{n}n^2 = 0,$$
$$\sigma P^2 - R(1+c) + \frac{q(R-P)^2}{n}\left[\frac{1}{2dP} + \frac{R-P}{2dP^2+qR}\right] = 0.$$

From the last equation we can obtain n in terms of P and the parameters of the model:

$$n = \frac{q(R-P)^2\left[\frac{1}{2dP} + \frac{R-P}{2dP^2+qR}\right]}{R(1+c) - \sigma P^2} \equiv \psi(P). \tag{85}$$

We can similarly obtain an expression for b in terms of P and the parameters of the model using the equilibrium price function $P = R + (b-c)n$, which implies that

$$b = \frac{P-R}{n} + c = \frac{P-R+cn}{n} = \frac{P-R+c\psi(P)}{\psi(P)}. \tag{86}$$

Now, substitute these expressions for n and b back into the banks' first-order condition (80) in order to obtain a fixed-point equation that involves only P as an endogenous variable, from which we can solve for the equilibrium price P:

$$\begin{aligned}
-q + \frac{qR}{P} &= 2dn(1+b), \\
\frac{qR}{P} &= 2d\psi(P)\left[\frac{P-R+c\psi(P)}{\psi(P)} + 1\right] + q, \\
\frac{qR}{P} &= 2d\psi(P)\left[\frac{P-R+(1+c)\psi(P)}{\psi(P)}\right] + q.
\end{aligned}$$

Multiply the last equation with P and rearrange to obtain

$$2d\left[P-R+(1+c)\psi(P)\right] + qP - qR = 0,$$
$$-2dP(R-P) - q(R-P) + 2d(1+c)P\psi(P) = 0.$$

Rearrange the last equation and substitute for $\psi(P)$ from (85):

$$
\begin{aligned}
2d(1+c)P\psi(P) &= (R-P)(2dP+q) \\
2d(1+c)P\frac{q(R-P)^2\left[\frac{1}{2dP}+\frac{R-P}{2dP^2+qR}\right]}{R(1+c)-\sigma P^2} &= (R-P)(2dP+q) \\
2d(1+c)qP(R-P)\left[\frac{1}{2dP}+\frac{R-P}{2dP^2+qR}\right] &= \left[R(1+c)-\sigma P^2\right](2dP+q) \\
(1+c)q(R-P)\left[1+\frac{2dP(R-P)}{2dP^2+qR}\right] &= \left[R(1+c)-\sigma P^2\right](2dP+q) \\
(1+c)q(R-P)\left[\frac{2dP^2+qR+2dPR-2dP^2}{2dP^2+qR}\right] &= \left[R(1+c)-\sigma P^2\right](2dP+q) \\
(1+c)q(R-P)\left[\frac{R(2dP+q)}{2dP^2+qR}\right] &= \left[R(1+c)-\sigma P^2\right](2dP+q).
\end{aligned}
$$

Lastly, simplifying $2dP+q$ from both sides and rearranging yields

$$
(1+c)q(R-P)R-(2dP^2+qR)\left[R(1+c)-\sigma P^2\right]=0. \tag{87}
$$

Substitute β for $R(1+c)$ from equation 43, and then expand this equation to obtain a polynomial equation in P:

$$
\begin{aligned}
q(R-P)\beta-(2dP^2+qR)\left[\beta-\sigma P^2\right] &= 0 \\
qR\beta-q\beta P-2d\beta P^2+2d\sigma P^4-qR\beta+qR\sigma P^2 &= 0 \\
2d\sigma P^4+(qR\sigma-2d\beta)P^2-q\beta P &= 0.
\end{aligned}
$$

Because we are interested in non-zero and positive equilibrium price for the illiquid asset, divide this last equation by P to obtain a cubic equation in P:

$$
2d\sigma P^3+[qR\sigma-2d\beta]P-q\beta=0. \tag{88}
$$

It is easy to show that this cubic equation has only one real root and two complex conjugate roots. The only real root can easily be obtained using Vieta's substitution for cubic equations.

8.3 Proofs for the Online Appendix

Proposition 10. *Risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium* (n, b, P, $1-\gamma$, $(1-\gamma)n$)*, partial regulation equilibrium* (n^*, b^*, P^*, $1-\gamma^*$, $(1-\gamma^*)n^*$)*, and complete regulation equilibrium* (n^{**}, b^{**}, P^{**}, $1-\gamma^{**}$, $(1-\gamma^{**})n^{**}$) *compare as follows:*

1. *Risky investment levels:* $n > n^{**} > n^*$
2. *Liquidity ratios:* $b^{**} > b > b^*$
3. *Financial stability measures*
   (a) *Price of assets in the bad state:* $P^{**} > P^* > P$

(b) *Fraction of assets sold:* $1-\gamma > 1-\gamma^* > 1-\gamma^{**}$

(c) *Total fire sales:* $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

Proof. Proof of this proposition is established through a series of lemmas below. □

Lemma 8. $P^{**} > P^* > P$

Proof. **Part 1:** $P^* > P$. First, note that we obtain the competitive equilibrium price of assets in the main text as:

$$P = \frac{qR(1+c)}{R-1+q} = \frac{\beta}{R\sigma}, \tag{89}$$

using the definitions of σ, β from (39) and (43). Now, take the cubic equation given by (88) and divide it by $R\sigma$ to obtain:

$$\frac{2d}{R}P^{*3} + \left[q - 2d\frac{\beta}{R\sigma}\right] P^* - q\frac{\beta}{R\sigma} = 0 \tag{90}$$

Note that $\beta/R\sigma = P$, and substitute this into the equation above and manipulate:

$$\frac{2d}{R}P^{*3} + [q - 2dP]\, P^* - qP = 0, \tag{91}$$

$$\left(\frac{2d}{R}P^{*2} + q\right) P^* = (2dP^* + q)P. \tag{92}$$

From this last equivalence we can obtain the price ratios in these two cases as:

$$\frac{P}{P^*} = \frac{\frac{2d}{R}P^{*2} + q}{2dP^* + q} = \frac{2dP^{*2} + qR}{2dRP^* + qR} < 1, \tag{93}$$

The last inequality holds because $P^{*2} < RP^*$, which is in turn true since we must have $P^* < R$ in equilibrium. Therefore, $P^* > P$.

$P^* < R$ holds in equilibrium for the following reason: Assumption *Concavity* states that $P^* \leq R$, yet the equality cannot arise in equilibrium as $P^* = R$ implies $P = R$ as well due to (93). However $P < R$ holds due to the *Technology* assumption, $R - cq - 1 > 0$. Thus, we must have $P^* < R$.

Part 2: $P^{**} > P^*$. Note that, in the solution to the complete regulation case in Section 8.1, we obtain

$$P^{**} = \sqrt{\frac{qR^2(1+c)}{R-1+q}} = \sqrt{\frac{\beta}{\sigma}}, \tag{94}$$

where σ, β are defined by (39) and (43). We start from the cubic equation obtained in the solution for the partial case that gives P^*. We repeat this cubic equation below for convenience:

$$2d\sigma P^{*3} + [qR\sigma - 2d\beta]P^* - q\beta = 0. \tag{95}$$

Divide this equation by σ to obtain:

$$2dP^{*3} + \left[qR - 2d\frac{\beta}{\sigma}\right] P^* - q\frac{\beta}{\sigma} = 0. \tag{96}$$

Note that $\beta/\sigma = P^{**2}$, and substitute this into the equation above and manipulate:

$$2dP^{*3} + \left[qR - 2dP^{**2}\right] P^* - qP^{**2} = 0, \tag{97}$$
$$2dP^{*3} + qRP^* - 2dP^{**2}P^* - qP^{**2} = 0, \tag{98}$$
$$(2dP^{*2} + qR)P^* = (2dP^* + q)P^{**2}. \tag{99}$$

Multiply both sides of this equation by P^* to obtain:

$$(2dP^{*2} + qR)P^{*2} = (2dP^{*2} + qP^*)P^{**2}. \tag{100}$$

From this last equivalence we can obtain the square of the price ratios in the two cases as:

$$\left(\frac{P^{**}}{P^*}\right)^2 = \frac{2dP^{*2} + qR}{2dP^{*2} + qP^*} > 1, \tag{101}$$

because we have $R > P^*$ in equilibrium, as explained above in Part 1. Therefore, $P^{**} > P^*$. □

Lemma 9. $b^{**} > b > b^*$

Proof. **Part 1:** $b^{**} > b$. Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (25) and (69) as:

$$b = \frac{cq - 2d\frac{R}{\tau+1}}{2d\frac{R}{\tau+1} + q}, \tag{102}$$
$$b^{**} = \frac{cq - 2d\frac{R}{(\tau^{**}+1)(\tau^{**}+2)}}{2d\frac{R}{(\tau^{**}+1)(\tau^{**}+2)} + q}, \tag{103}$$

where

$$\tau = \frac{R}{P} - 1 = \frac{R - 1 + q}{q(1+c)} - 1 \equiv \eta^2 - 1, \tag{104}$$
$$\tau^{**} = \frac{R}{P^{**}} - 1 = \sqrt{\frac{R - 1 + q}{q(1+c)}} - 1 \equiv \eta - 1. \tag{105}$$

Therefore, $(\tau^{**} + 1)(\tau^{**} + 2) = \eta(\eta + 1) > \tau + 1 = \eta^2 \implies b^{**} > b$.

Part 2: $b > b^*$. Comparing the liquidity ratios under competitive equilibrium (b), given by (25), and under the partial regulation case (b^*), given by (73), we see that they have the same functional form, the only difference is τ versus τ^*. Note that because $\tau^* = R/P^* - 1$ and $\tau = R/P - 1$, and $P^* > P$ by Lemma 8, we have that $\tau > \tau^*$. Therefore, in order to show that $b > b^*$, all we need to show is that b is increasing in τ:

$$\frac{db}{d\tau} = \frac{qc[2dR + q(\tau + 1)] - q[qc(\tau + 1) - 2dR]}{[2dR + q(\tau + 1)]^2} = \frac{2dRq(c + 1)}{[2dR + q(\tau + 1)]^2} > 0. \tag{106}$$

□

Lemma 10. $n > n^{**} > n^*$

Proof. **Part 1:** $n > n^{**}$. We will use $n^{**} = \frac{P^{**}\tau^{**}}{c-b^{**}}$, which is the counterpart of (24), and $c - b^{**} = \frac{2dR(1+b^{**})}{q(\tau^{**}+1)(\tau^{**}+2)}$, which we derived earlier by (68). Plugging the latter into the former and using $P^{**} = \frac{R}{\tau^{**}+1}$ we get

$$n^{**} = R\frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)(\tau^{**}+2)}{2dR(1+b^{**})} = \frac{q}{2d}\frac{\tau^{**}(\tau^{**}+2)}{1+b^{**}}. \tag{107}$$

For the competitive equilibrium, similarly, plug equation (7.1) into (24) and replace P as $R/(\tau+1)$ from (23) to write the equilibrium risky investment level as $n = \frac{q}{2d}\frac{\tau}{1+b}$. Thus, we have

$$\begin{aligned} n - n^{**} &= \frac{q}{2d}\left(\frac{\tau}{1+b} - \frac{\tau^{**}(\tau^{**}+2)}{1+b^{**}}\right), \\ &= \frac{q}{2d}\left(\frac{\eta^2-1}{1+b} - \frac{(\eta-1)(\eta+1)}{1+b^{**}}\right), \\ &= \frac{q}{2d}\left(\frac{\eta^2-1}{1+b} - \frac{\eta^2-1}{1+b^{**}}\right) > 0, \end{aligned}$$

where we use equations (104) and (105) to replace τ's in terms of η's. The sign is positive because $b^{**} > b$ by Lemma 9. Therefore, $n > n^{**}$.

Part 2: $n^{**} > n^*$. For the second part of this lemma, we will use the fact that $P^{**} > P^*$ as proven by Lemma 8. Take equation (101) that gives the square of the price ratios in these two cases and replace P^* and P^{**} according to relationship $P = \frac{R}{\tau+1}$ to obtain:

$$\left(\frac{P^{**}}{P^*}\right)^2 = (\frac{\tau^*+1}{\tau^{**}+1})^2 = \frac{2dR + q(\tau^*+1)^2}{2dR + q(\tau^*+1)}. \tag{108}$$

Previously, we obtained the closed-form solutions for n^{**} and n^* by (70) and (74) as follows:

$$n^{**} = \frac{\tau^{**}}{\tau^{**}+1}\frac{2dR + q(\tau^{**}+1)(\tau^{**}+2)}{2d(1+c)},$$

$$n^* = \frac{\tau^*}{\tau^*+1}\frac{2dR + q(\tau^*+1)}{2d(1+c)}.$$

Consider the difference

$$n^{**} - n^* = \frac{1}{2d(1+c)}\left(\frac{\tau^{**}(\tau^*+1)[2dR + q(\tau^{**}+1)(\tau^{**}+2)] - \tau^*(\tau^{**}+1)[2dR + q(\tau^*+1)]}{2d(1+c)(\tau^*+1)(\tau^{**}+1)}\right).$$

Remember that the terms, τ, τ^*, and τ^{**}, represent the total amount of risky assets sold at the fire sale prices, and hence, they are positive (please see (23) for the definition of τ). Therefore, we can focus on the numerator of the difference to determine the sign of the difference. The numerator can be simplified as follows

$$2dR\underbrace{[\tau^{**}(\tau^*+1)-\tau^*(\tau^{**}+1)]}_{\tau^{**}-\tau^*}+q(\tau^*+1)(\tau^{**}+1)\underbrace{[\tau^{**}(\tau^{**}+2)-\tau^*]}_{(\tau^{**}+1)^2-(\tau^*+1)} \tag{109}$$

From (108) we have that

$$(\tau^{**}+1)^2=(\tau^*+1)^2\frac{2dR+q(\tau^*+1)}{2dR+q(\tau^*+1)^2}. \tag{110}$$

Plug this equation into the last part of numerator to get

$$(\tau^{**}+1)^2-(\tau^*+1)=(\tau^*+1)\left((\tau^*+1)\frac{2dR+q(\tau^*+1)}{2dR+q(\tau^*+1)^2}-1\right)=\frac{(\tau^*+1)2dR\tau^*}{2dR+q(\tau^*+1)^2}.$$

Finally, plug the last equivalence back into (109) to obtain

$$2dR[\tau^{**}-\tau^*]+\frac{q\tau^*(\tau^{**}+1)(\tau^*+1)\tau^*2dR}{2dR+q(\tau^*+1)^2}. \tag{111}$$

We will show that (111) is positive. The second additive term in (111) is clearly positive, while the first term is negative because $\tau^*>\tau^{**}$, as implied by Lemma 8. Thus, we show that the second part is larger than the absolute value of the first part, $2dR(\tau^*-\tau^{**})$:

$$\begin{aligned}
\frac{2dRq\tau^*(\tau^*+1)^2(\tau^{**}+1)}{2dR+q(\tau^*+1)^2} &> 2dR(\tau^*-\tau^{**})\\
\frac{q\tau^*(\tau^*+1)^2(\tau^{**}+1)}{2dR+q(\tau^*+1)^2} &> (\tau^*-\tau^{**})\\
q\tau^*(\tau^*+1)^2(\tau^{**}+1) &> 2dR(\tau^*-\tau^{**})+q(\tau^*-\tau^{**})(\tau^*+1)^2\\
(\tau^*+1)^2q\underbrace{[\tau^*(\tau^{**}+1)-\tau^*+\tau^{**}]}_{=\tau^{**}(\tau^*+1)} &> 2dR(\tau^*-\tau^{**}).
\end{aligned}$$

Thus, the inequality can also be stated as

$$(\tau^*+1)^2q\tau^{**}(\tau^*+1)-2dR(\tau^*-\tau^{**})>0. \tag{112}$$

We expand the left-hand side of this equation:

$$\begin{aligned}
(\tau^*+1)^3q\tau^{**}-2dR(\tau^*-\tau^{**}) &= (\tau^*+1)^3q\tau^{**}-2dR[(\tau^*+1)-(\tau^{**}+1)]\\
&\Longrightarrow \frac{\tau^*+1}{\tau^{**}+1}(\tau^*+1)^2q\tau^{**}+2dR-2dR\frac{\tau^*+1}{\tau^{**}+1}\\
&= \frac{\tau^*+1}{\tau^{**}+1}[(\tau^*+1)^2q\tau^{**}-2dR]+2dR.
\end{aligned}$$

Let us focus on $q\tau^{**}(\tau^*+1)^2-2dR$ because the remaining terms are positive. For that first we will get rid of $2dR$ by replacing it with a term in terms of τ^*'s and η's. We have η^2 from (110) as follows

$$\eta^2 = (\tau^{**}+1)^2 = (\tau^*+1)^2 \frac{2dR + q(\tau^*+1)}{2dR + q(\tau^*+1)^2}. \tag{113}$$

We also derive η^2 from (105) as follows

$$\tau^{**} = \eta - 1 \implies \tau^{**}(\tau^{**}+2) = \eta^2 - 1 \implies \eta^2 = \tau^{**}(\tau^{**}+2) + 1.$$

Equating these two derivations gives us $2dR$ in terms of τ^*'s and η's:

$$\begin{aligned}
\eta^2 = \tau^{**}(\tau^{**}+2)+1 &= \frac{2dR(\tau^*+1)^2 + q(\tau^*+1)^3}{2dR + q(\tau^*+1)^2} \\
2dR\eta^2 + q(\tau^*+1)^2\eta^2 &= 2dR(\tau^*+1)^2 + q(\tau^*+1)^3 \\
2dR[\eta^2 - (\tau^*+1)^2] &= q(\tau^*+1)^3 - q(\tau^*+1)^2\eta^2 = q(\tau^*+1)^2[\tau^*+1-\eta^2],
\end{aligned}$$

from which we can obtain

$$2dR = \frac{q(\tau^*+1)^2[\tau^*+1-\eta^2]}{\eta^2 - (\tau^*+1)^2}. \tag{114}$$

Finally, we plug this expression back into $q\tau^{**}(\tau^*+1)^2 - 2dR$, and show that this term is positive

$$\begin{aligned}
q\tau^{**}(\tau^*+1)^2 - 2dR &= q\tau^{**}(\tau^*+1)^2 - \frac{q(\tau^*+1)^2[\tau^*+1-\eta^2]}{\eta^2-(\tau^*+1)^2} \\
&= q(\tau^*+1)^2\left(\tau^{**} - \frac{\tau^*+1-\eta^2}{\eta^2-(\tau^*+1)^2}\right) \\
&= q(\tau^*+1)^2\left(\tau^{**} + \frac{\eta^2-(\tau^*+1)}{\eta^2-(\tau^*+1)^2}\right) \\
&= q(\tau^*+1)^2\left(\eta - 1 + \frac{\eta^2-(\tau^*+1)}{\eta^2-(\tau^*+1)^2}\right) > 0.
\end{aligned}$$

The sign is positive because $\frac{\eta^2-(\tau^*+1)}{\eta^2-(\tau^*+1)^2} > 1$ and $\eta - 1 = \tau^{**} > 0$, as given by definition (105). □

Lemma 11. $1-\gamma > 1-\gamma^* > 1-\gamma^{**}$

Proof.

$$1-\gamma = \frac{c-b}{P} \text{ together with } b^{**} > b^* \text{ and } P^{**} > P^* \implies 1-\gamma^* > 1-\gamma^{**}$$

To obtain $(1-\gamma) > (1-\gamma^*)$, we can equivalently show that $\frac{\frac{c-b}{P}}{\frac{c-b^*}{P^*}} > 1$.

Given $b = \frac{qc(\tau+1)-2dR}{2dR+q(\tau+1)} \implies c-b = \frac{2dR(1+c)}{2dR+q(\tau+1)}$, and similarly $c - b^* = \frac{2dR(1+c)}{2dR+q(\tau^*+1)}$

$$\frac{\frac{c-b}{P}}{\frac{c-b^*}{P^*}} = \frac{c-b}{c-b_1}\frac{P}{P^*} = \frac{2dP^*+q}{2dP+q}\frac{P}{P^*}\frac{P^*}{P} > 1.$$

The last inequality is due to $P^* > P$ by Lemma 8. □

Lemma 12. $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

Proof. Given that the demand function for risky assets in the interim period is downward sloping (continuous and differentable as well), the prices will be informative about the amount of fire sales. Hence, we can use the results in Lemma 8 to prove this lemma:

$$(1-\gamma)n = \tau = \frac{R}{P} - 1 \text{ and } P^{**} > P^* > P \implies (1-\gamma^{**})n^{**} < (1-\gamma^*)n^* < (1-\gamma)n$$

□

Proposition 11. *Bank balance sheet sizes across different regimes are as follows:*

$$n(1+b) = n^{**}(1+b^{**}) > n^*(1+b^*)$$

Proof. Using the closed-form solutions in Sections 7.1 and 8.1, we can write the bank size under the competitive equilibrium n^{**} and n^* by (70) and (74) as follows:

$$\begin{aligned} n(1+b) &= \frac{\tau}{\tau+1}\frac{2dR+q(\tau+1)}{2d(1+c)}\frac{q(\tau+1)(1+c)}{2dR+q(\tau+1)}, \\ &= \frac{\tau}{\tau+1}\frac{q(\tau+1)}{2d}, \\ &= \frac{q\tau}{2d}. \end{aligned}$$

$$\begin{aligned} n^{**}(1+b^{**}) &= \frac{\tau^{**}}{\tau^{**}+1}\frac{2dR+q(\tau^{**}+1)(\tau^{**}+2)}{2d(1+c)}\frac{q(\tau^{**}+1)(\tau^{**}+2)(1+c)}{2dR+q(\tau^{**}+1)(\tau^{**}+2)}, \\ &= \frac{q\tau^{**}(\tau^{**}+1)}{2d}. \end{aligned}$$

Note that $\tau = \eta^2 - 1$ and $\tau^{**} = \eta - 1 \implies \tau^{**}(\tau^{**}+2) = (\eta-1)(\eta+1) = \eta^2 - 1 = \tau$. Thus, $n(1+b) = n^{**}(1+b^{**})$.

Lastly, $b^{**} > b > b^*$ and $n > n^{**} > n^*$ together imply that $n(1+b) > n^*(1+b^*)$, that is, the bank balance sheet size is smallest under partial regulation. □