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Appendix to

"Optimal Monetary Policy with Durable Consumption Goods"

Abstract

In this appendix, we show how we derive a second order approximation to the social welfare function of our model.

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1 The Welfare Function

To provide a normative assessment of alternative monetary policy choices, we measure social welfare as the unconditional expectation of average household lifetime utility:

$$SW = \mathcal{E}\int_0^1 \left[\sum_{i=0}^{\infty} \beta^t\, \mathbb{W}_t\,(h)\right]\, dh \tag{1}$$

where the term in large brackets is the discounted lifetime utility functional of household h presented in Section 3.2 of our paper (see equations 11 and 12). In this appendix, we follow the seminal analysis of Rotemberg and Woodford (1997) in deriving the second-order approximation to each component of the social welfare function and computing its deviation from the welfare of the Pareto-optimal equilibrium under flexible wages and prices.

It is useful to decompose household $h's$ *period* utility function $\mathbb{W}_t\,(h)$,as follows:

$$\mathbb{W}_t\,(h) = \mathbb{W}_t^s(h) + \mathbb{W}_t^m\,(h) + \mathbb{M}\left(\tfrac{M_t(h)}{P_{st}}\right)$$

$$\mathbb{W}_t^S\,(h) = \mathbb{S}(C_t\,(h)\,) + \mathbb{Z}\,(N_{st}\,(h)) \tag{2}$$

$$\mathbb{W}_t^m\,(h) = \mathbb{U}(\widetilde{D}_t\,(h)) + \mathbb{V}\,(N_{mt}\,(h))$$

where $\mathbb{W}_t^S\,(h)$ indicates the household's period utility associated with non-durables consumption and service-sector employment, while $\mathbb{W}_t^m\,(h)$ denotes the period utility associated with durables consumption and manufacturing employment. Accordingly, given our assumption that welfare losses due to fluctuations in real money balances are arbitrarily small, social welfare may be regarded as comprised of a "nondurables (and service-sector employment) component" and a "durables (and manufacturing employment) component":

$$SW = SW^S + SW^M = \mathcal{E}\int_0^1 \left[\sum_{i=0}^{\infty} \beta^t\, \mathbb{W}_t^S\,(h)\right]\, dh + \mathcal{E}\int_0^1 \left[\sum_{t=0}^{\infty} \beta^t\, \mathbb{W}_t^M\,(h)\right]\, dh \tag{3}$$

We begin by taking a second order approximation to the nondurables component SW^s, since the derivation closely parallels that in Erceg, Henderson, and Levin 2000 (aside from the differences introduced by the contract structure); we then proceed to approximate the durables component. We also present results for the case in which social welfare is taken to be the "time zero" conditional expectation of the discounted utility flow averaged across households (in which case we denote social welfare by SW_0, and substitute the conditional expectation operator $\mathcal{E}_0$). Our derivations express the welfare loss attributable to each component (relative to the Pareto optimal equilibrium) as a percent of steady state output.

1.1 Non-durables Component of Social Welfare

The non-durables component of social welfare depends on the expected period utility associated with non-durables averaged across households (henceforth, we refer to this concept of period utility averaged across households as "period welfare" in non-durables). Because there is perfect consumption risk-sharing, period welfare in non-durables may be written as:

$$\mathbb{W}_t^S = \int_0^1 \mathbb{W}_t^S(h)\,dh = \mathbb{S}(C_t) + \int_0^1 \mathbb{Z}(N_{st}(h))\,dh = \mathbb{S}(C_t) + \mathcal{E}_h \mathbb{Z}(N_{st}(h)) \tag{4}$$

This expression is essentially identical to period welfare in the one sector model of Erceg, Henderson, and Levin 2000 (henceforth EHL) that appears in equation 21 of their paper.[1] Accordingly, following the steps outlined in Appendix B of that paper, the deviation of the period welfare function in non-durables from its Pareto optimal level $\mathbb{W}_t^{S*}$ may be approximated to second order as:

[1] The slight difference with EHL reflects that in our present analysis, we assume that there is a government spending shock (to non-durables) rather than a taste shock to the marginal utility of consumption, and we also omit consideration of a taste shock to the marginal disutility of work.

$$\left(\mathbb{W}_t^S - \mathbb{W}_t^{S*}\right) \approx -\tfrac{1}{2} S_c Y_s \left(\mu^s_{mrs} + \mu^s_{mpl}\right) g^2_{st}$$

$$+\tfrac{1}{2} N_s^2 Z_{N_s N_s} var_h n_{st}(h) \tag{5}$$

$$+\tfrac{1}{2} N_s Z_{N_s} \left(\tfrac{\vartheta_w}{1+\vartheta_w} var_h n_{st}(h) + \tfrac{\vartheta_P}{1+\vartheta_P}\tfrac{1}{1-\alpha_S} var_f y_{st}(f)\right)$$

where g_{st} denotes the output gap in non-durables, $n_{st}(h)$ indicates the (logarithmic) percent deviation from steady state of the labor hours of members of household h that work in non-durables, and $var_h n_{st}(h)$ indicates the cross-sectional dispersion of $n_{st}(h)$ around the cross-sectional average of $\mathcal{E}_h n_{st}(h)$.[2] Similarly, $y_{st}(f)$ indicates the percent deviation from steady state of the output of non-durables firm f, and $var_f y_{st}(f)$ indicates the cross-sectional dispersion of $y_{st}(f)$ around the cross-sectional average of $\mathcal{E}_f y_{st}(f)$. The coefficients $\left(\mu^s_{mrs} + \mu^s_{mpl}\right)$ may be interpreted as the sum of the absolute values of the slopes of the marginal rate of substitution and the marginal product of labor schedules in non-durables with respect to output.[3] As in EHL, the first term of equation (5) captures welfare losses due to deviations of non-durables output from potential, the second term reflects costs of cross-sectional dispersion in hours worked in non-durables that arise due to an increasing marginal disutility of work, while the final term reflects productive inefficiencies in non-durables associated with cross-sectional dispersion in that sector's production and employment.[4]

The labor demand equation directly implies that:

[2] In this appendix, we use lower case letters to denote the logarithimic percent deviation of a variable from its steady state level.

[3] The form of equation 5 is virtually identical to that in EHL, equation 22, with the only differences that i) steady state output Y_s appears in 5rather than the level of consumption C (as consumption does not equal output in our model due to the government spending wedge), and ii) there is a slight change in the definition of the $\left(\mu^s_{mrs} + \mu^s_{mpl}\right)$ coefficients. In our model, the relationship between this sum and the structural coefficients is given by $\mu^s_{mrs} + \mu^s_{mpl} = (\sigma_s \frac{1-\alpha_s}{1-S^s_G} + \alpha_s + \chi_s \ell_s)/(1-\alpha_s)$. In the EHL specification, the $\frac{1}{1-S^s_G}$ coefficient is replaced the ratio of (steady state) consumption to consumption net of the taste shock (ℓ_C in their notation).

[4] Note that the marginal disutility of labor in non-durables is positive and increasing (-$Z_{N_s} > 0$ and -$Z_{N_s N_s} > 0$), so that all three terms on the right hand side of equation (5) are negative.

$$var_h n_{st}(h) = \left(\frac{1+\theta_{W_S}}{\theta_{W_S}}\right)^2 var_h \ln W_{st}(h) \tag{6}$$

Thus, cross-sectional dispersion in hours worked in non-durables across the members of different households varies directly with cross-sectional wage dispersion, with the sensitivity of the former to wage dispersion rising as labor services become closer to perfect substitutes (i.e., as the wage markup θ_W declines toward zero). Given our assumption of fixed-duration four quarter ("Taylor-style") wage contracts, the wage dispersion term can be expressed as:

$$var_h \ln W_{st}(h) = \frac{1}{4}\sum_{j=0}^{3} \ln\left(\tilde{W}_{s,t-j} \,/\, W_{st}\right)^2 \tag{7}$$

where $\tilde{W}_{s,t-j}$ indicates the contract wage signed by households at period $t-j$ (for $j = 0, ..., 3$). The simple form of this wage dispersion term reflects that there are effectively only four cohorts of households (since each household that reoptimizes its wage at the same time chooses the same contract wage). Similarly, using the demand curve for each monopolistic producer, cross-sectional output dispersion can be written in terms of dispersion in relative prices across the four cohorts of producers:

$$var_f y_{st}(f) \approx \left(\frac{1+\theta_{P_S}}{\theta_{P_S}}\right)^2 var_h P_{st}(f) = \frac{1}{4}\sum_{j=0}^{3} \ln\left(\tilde{P}_{s,t-j} \,/\, P_{st}\right)^2 \tag{8}$$

where $\tilde{P}_{s,t-j}$ indicates the contract price signed by firms at period $t-j$ (for $j = 0,...,3$).

The approximate period welfare loss function is obtained by substituting equations (6),(7), and (8) into the period welfare function for non-durables, and then subtracting an analogous approximation for period welfare in the Pareto-optimal equilibrium. Upon taking the conditional (time zero) expectation of the resulting expression and scaling by $Y\mathbb{S}_C$, we obtain the conditional expected *period* welfare loss in non-durables as a percent of steady state output:

$$\frac{1}{Y\mathbb{S}_C}\mathcal{E}_0\left(\mathbb{W}_t^S - \mathbb{W}_t^{S*}\right) \approx -\tfrac{1}{2}\psi_s\left(\mu_{mrs}^s + \mu_{mpl}^s\right)\mathcal{E}_0 g_{st}{}^2$$

$$-\tfrac{1}{8}\psi_s\left(\tfrac{1+\theta_{P_S}}{\theta_{P_S}}\right)\sum_{j=0}^{3}\mathcal{E}_0 \ln\left(\tilde{P}_{s,t-j}\,/\,P_{st}\right)^2 \tag{9}$$

$$-\tfrac{1}{8}\psi_s\left(\tfrac{1+\theta_{W_S}}{\theta_{W_S}}\right)(1-\alpha_s)\left(1+\left(\tfrac{1+\theta_{W_S}}{\theta_{W_S}}\right)\chi_s\ell_s\right)\sum_{j=0}^{3}\mathcal{E}_0 \ln\left(\tilde{W}_{s,t-j}\,/\,W_{st}\right)^2$$

The derivation of (9) utilizes the steady state first order condition for labor supply that $Z_{N_s}N_s = -(1-\alpha_S)S_cY_S$, and that $Z_{N_sN_s}N_s^2 = Z_{N_s}N_s\chi_s\left(\frac{N_s}{N-N_s}\right) = -(1-\alpha_S)S_cY_SN_s\chi_s\ell_s$ (recall also that $\psi_s = \frac{Y_S}{Y}$ is the service sector's share of total output). The conditional social welfare loss in non-durables is simply the discounted sum of the conditional expected period welfare losses, so that:

$$\frac{1}{Y\mathbb{S}_C}\left(SW_0^S - SW_0^{S*}\right) = \frac{1}{Y\mathbb{S}_C}\sum_{t=0}^{\infty}\beta^t\mathcal{E}_0\left(\mathbb{W}_t^S - \mathbb{W}_t^{S*}\right) \tag{10}$$

Finally, the unconditional social welfare loss in durables – which is the metric utilized in the analysis of the paper – is obtained by simply taking the unconditional expectation of each side of (10) (since the unconditional social welfare function $SW^S = \mathcal{E}SW_0^S$). The unconditional period welfare loss function is isomorphic to (9), except that the terms involving conditional second moments are replaced by unconditional variances.[5]

1.2 Durables Component of Social Welfare

We refer to $\mathbb{W}_t^m$ as "period welfare" in durables (it represents an unweighted average of household period utility derived from consuming durable goods, and from the employment of its members in the manufacturing sector). Recalling that there is perfect risk-sharing for the consumption of durable goods, the period welfare attributable to durables may be expressed as:

[5] As shown in Appendix B of EHL, the unconditional variance of each variable is equal to its unconditional second moment (at least to a second order approximation).

$$\mathbb{W}_t^M = \int_0^1 \mathbb{W}_t^M(h)\,dh = \mathbb{U}(\widetilde{D}_t) + \int_0^1 \mathbb{V}(N_{mt}(h))\,dh = \mathbb{U}(\widetilde{D}_t) + \mathcal{E}_h \mathbb{V}(N_{mt}(h)) \tag{11}$$

A second order approximation of $\sum_{t=0}^{\infty} \beta^t\, \mathbb{W}_t^M$ (the discounted stream of period welfare in durables) yields:

$$\begin{aligned} \textstyle\sum_{t=0}^{\infty} \beta^t\, \mathbb{W}_t^M \approx \sum_{t=0}^{\infty} \beta^t \{ & DU_D \hat{D}_t \\ & + \tfrac{1}{2} D^2 U_{DD} \hat{D}_t^2 - \tfrac{1}{2}\phi DU_D(\hat{D}_{t+1} - \hat{D}_t)^2 \\ & + N_m V_{N_m} \mathcal{E}_h \hat{N}_{mt}(h) + \tfrac{1}{2} N_m^2 V_{N_m N_m} \mathcal{E}_h \hat{N}_{mt}^{\,2}(h) \} \end{aligned} \tag{12}$$

where $\hat{D}_t(h)$ denotes the *arithmetic* percent deviation of the stock of durables from its steady state level, and $\hat{N}_{mt}(h)$ the arithmetic percent deviation from steady state of the hours worked by each member of household h that works in manufacturing (for expositional simplicity, we ignore constant terms that arise in the approximation). Using the fact that the arithmetic percent deviation from steady state of a generic variable "X_t" may be written to a second order approximation as $\widehat{X_t} = \frac{X_t - X}{X} \approx x_t + \frac{1}{2}x_t^2$, where "$x_t$" is the logarithmic percent deviation from steady state, equation (12) may be reexpressed as:

$$\begin{aligned} \textstyle\sum_{i=0}^{\infty} \beta^i\, \mathcal{E}_h \mathbb{W}_{t+i}^m(h) \approx \sum_{i=0}^{\infty} \beta^i \{ & DU_D d_{t+i} \\ & + \tfrac{1}{2}(DU_D + D^2 U_{DD}) d_{t+i}^2 - \tfrac{1}{2}\phi DU_D (d_{t+i+1} - d_{t+i})^2 \\ & + N_m V_{N_m} \mathcal{E}_h n_{m,t+i}(h) + \tfrac{1}{2}(N_m V_{N_m} + N_m^2 V_{N_m N_m}) \mathcal{E}_h n_{m,t+i}^{\,2}(h) \} \end{aligned} \tag{13}$$

Given that the structure for producing both manufactured goods and the composite labor aggregate in manufacturing is isomorphic to that in the one sector model of EHL, the cross-sectional average percent deviation in labor hours in manufacturing $\mathcal{E}_h n_{m,t}(h)$ has the same form as in

EHL (see equation B13 of their appendix). Thus:

$$\mathcal{E}_h n_{mt}(h) \approx \frac{1}{1-\alpha_m}(y_{mt}-a_{mt}) + \frac{1}{2}\left(\frac{\theta_{Pm}}{1+\theta_{Pm}}\right)\frac{1}{1-\alpha_m} var_f y_m(f) - \frac{1}{2}\left(\frac{1}{1+\theta_{Wm}}\right) var_h n_m(h). \tag{14}$$

while the second moment term $\mathcal{E}_h n_{mt}(h)^2$ can be related to the cross-sectional dispersion in manufacturing hours worked by:

$$\mathcal{E}_h n_{mt}(h)^2 \approx \left(\frac{1}{1-\alpha_m}(y_{mt}-a_{mt})\right)^2 + var_h n_{mt}(h). \tag{15}$$

Substituting (14) and (15) into equation (13) yields:

$$\begin{aligned}
\sum_{t=0}^{\infty}\beta^t \mathbb{W}_t^m &\approx \sum_{t=0}^{\infty}\beta^t DU_D d_t + \sum_{t=0}^{\infty}\beta^t N_m V_{N_m}\frac{1}{1-\alpha_m}(y_{mt}-a_{mt}) \\
&+\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\{(DU_D + D^2U_{DD})d_t^2 - \frac{1}{2}\phi DU_D(d_{t+1}-d_t)^2\} \\
&+\frac{1}{2}\left(N_m V_{N_m} + N_m^2 V_{N_m N_m}\right)\sum_{t=0}^{\infty}\beta^t\left(\frac{1}{1-\alpha_m}(y_{mt}-a_{mt})\right)^2 \\
&+\frac{1}{2}N_m V_{N_m}\frac{1}{1-\alpha_m}\left(\frac{\theta_{Pm}}{1+\theta_{Pm}}\right)\sum_{t=0}^{\infty}\beta^t var_f y_{mt}(f) \\
&+\frac{1}{2}\left(\left(\frac{\theta_{Wm}}{1+\theta_{Wm}}\right)N_m V_{N_m} + N_m^2 V_{N_m N_m}\right)\sum_{t=0}^{\infty}\beta^t var_h n_{mt}(h)
\end{aligned} \tag{16}$$

We next use a second order approximation to the transition law for the stock of durables to substitute out for the linear terms in equation (16). This is an important step, because it allows us to confine our attention to first-order approximations to the model's behavioral equations in making welfare evaluations that are valid to second order. It is easy to show that a second order logarithmic approximation to the transition law can be written:

$$d_{t+1} \approx (1-\delta)\,d_t + \delta y_{m,t} + \xi_t \tag{17}$$

where:

$$\xi_t = \frac{\delta\,(1-\delta)}{2}\,(y_{mt} - d_t)^2 \tag{18}$$

Using equation (17) to substitute recursively for d_t, the linear terms in (16) (ignoring the shock a_{mt}, which does not depend on policy) can be expressed as:

$$\sum_{t=0}^{\infty} \beta^t DU_D d_t + \sum_{t=0}^{\infty} \beta^t N_m V_{N_m} \tfrac{1}{1-\alpha_m} y_{mt} \approx$$
$$\tfrac{DU_D}{1-\beta(1-\delta)} d_0 + \sum_{t=0}^{\infty} \beta^t \left(\tfrac{\beta U_D D\delta}{1-\beta(1-\delta)} + N_m V_{N_m} \tfrac{1}{1-\alpha_m} \right) y_{mt} + \sum_{t=0}^{\infty} \beta^t \left(\tfrac{\beta D U_D}{1-\beta(1-\delta)} \right) \xi_t \tag{19}$$

The coefficient multiplying the second term drops. This reflects that the steady state first order conditions may be expressed in the form:

$$\frac{\beta U_D}{1-\beta(1-\delta)} = \Lambda_m = \frac{-V_{N_m} N_m}{(1-\alpha_m) Y_m} \tag{20}$$

(and recalling that $Y_m = \delta D$ in the steady state). The two left-hand side terms equate the discounted utility stream from a durable to its shadow cost Λ_m, while the right-hand side terms equate this cost with the ratio of the marginal disutility of work to the marginal product of labor in manufacturing; alternatively, the household's marginal rate of substitution between leisure and consumption in durables equals the marginal product of labor. Our assumption of subsidies in production and employment clearly is crucial in ensuring that (20) holds, hence allowing the linear terms in (19) involving y_{mt} to drop.

We next consider the deviation of social welfare in durables from its Pareto-optimal level. Substituting (19), including the definition of ξ_t, into (16), and subtracting the welfare that would prevail at the Pareto-optimum (using an analogous approximation), we obtain:

$$\sum_{t=0}^{\infty}\beta^t\mathbb{W}_t^M-\sum_{t=0}^{\infty}\beta^t\,\mathbb{W}_t^{M*}\approx\left(\frac{DU_D}{1-\beta(1-\delta)}\right)(d_0-d_0^*)$$

$$+\sum_{t=0}^{\infty}\beta^t\left(\frac{\beta DU_D}{1-\beta(1-\delta)}\right)\frac{\delta(1-\delta)}{2}\{(y_{mt}-d_t)^2-(y_{mt}^*-d_t^*)^2\}$$

$$+\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\{(DU_D+D^2U_{DD})(d_t^2-d_t^{*2})-\phi DU_D(\Delta d_{t+1}^2-\Delta d_{t+1}^{*2})\}$$

$$+\frac{1}{2}\left(N_mV_{N_m}+N_m^2V_{N_mN_m}\right)\sum_{t=0}^{\infty}\beta^t\{\left(\frac{y_{mt}-a_{mt}}{1-\alpha_m}\right)^2-\left(\frac{y_{mt}^*-a_{mt}}{1-\alpha_m}\right)^2\} \tag{21}$$

$$+\frac{1}{2}N_mV_{N_m}\frac{1}{1-\alpha_m}\left(\frac{\theta_{Pm}}{1+\theta_{Pm}}\right)\sum_{t=0}^{\infty}\beta^t var_f y_{mt}(f)$$

$$+\frac{1}{2}\left(\left(\frac{\theta_{Wm}}{1+\theta_{Wm}}\right)N_mV_{N_m}+N_m^2V_{N_mN_m}\right)\sum_{t=0}^{\infty}\beta^t var_h n_{mt}(h)$$

We note that:

$$(y_{mt}-a_{mt})^2-(y_{mt}^*-a_{mt})^2=(y_{mt}-y_{mt}^*)^2+(y_{mt}-y_{mt}^*)(y_{mt}^*-a_{mt}) \tag{22}$$

$$=g_{mt}^2+2(1-a_m)g_{mt}l_{mt}^*$$

where l_{mt}^* is hours worked in manufacturing in the Pareto optimal equilibrium. Moreover, using the relations $N_m^2V_{N_mN_m}=N_mV_{N_m}\chi_m\left(\frac{N_m}{N-N_m}\right)=N_mV_{N_m}\chi_m\ell_m,\ D^2U_{DD}=-\sigma_mDU_D$, and using the steady state first order conditions given in equation (20), we may express the coefficients in (21) in terms of Λ_mY_m. Upon taking the "time zero" conditional expectation of both sides of equation (21), we obtain:

$$SW_0^M - SW_0^{M*} = \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{W}_t^M - \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{W}_t^{M*}$$

$$\begin{aligned}
&\approx \left(\frac{\Lambda_m Y_m}{\beta\delta}\right)(d_0 - d_0^*) + \left(\frac{1-\delta}{2}\right)\Lambda_m Y_m \sum_{t=0}^{\infty} \beta^t \{\mathcal{E}_0 (y_{mt} - d_t)^2 - \mathcal{E}_0 (y_{mt}^* - d_t^*)^2\} \\
&+\frac{1}{2}\left(\frac{1-\beta(1-\delta)}{\beta\delta}\right)\Lambda_m Y_m \sum_{t=0}^{\infty} \beta^t \{(1-\sigma_m)(\mathcal{E}_0 d_t^2 - \mathcal{E}_0 d_t^{*2}) - \phi(\mathcal{E}_0 \Delta d_{t+1}^2 - \mathcal{E}_0 \Delta d_{t+1}^{*2})\} \\
&-\frac{1}{2}\Lambda_m Y_m (1 + \chi_m \ell_m)\frac{1}{1-\alpha_m} \sum_{t=0}^{\infty} \beta^t \{\mathcal{E}_0 g_{mt}^2 + 2(1-\alpha_m)\mathcal{E}_0(g_{mt}, l_{mt}^*)\} \\
&-\frac{1}{2}\Lambda_m Y_m \left(\frac{\theta_{Pm}}{1+\theta_{Pm}}\right) \sum_{t=0}^{\infty} \beta^t \mathcal{E}_0 var_f y_{mt}(f) \\
&-\frac{1-\alpha_m}{2}\Lambda_m Y_m \{\left(\frac{\theta_{Wm}}{1+\theta_{Wm}}\right) + \chi_m \ell_m\} \sum_{t=0}^{\infty} \beta^t \mathcal{E}_0 var_h n_{mt}(h)
\end{aligned} \tag{23}$$

The linear term involving $(d_0 - d_0^*)$ drops because we assume that the initial value of the stock of durables is equal to its steady state value, while expressions analogous to (6), (7), and (8) can be used to solve out for expected cross-sectional output dispersion and dispersion in household labor hours in terms of the conditional second moments of relative prices and wages in manufacturing. Moreover, we normalize the relative price of manufactured goods in terms of service goods to unity, i.e., $\Lambda_m = S_c$ (by appropriate choice of the scaling parameter in the utility function σ_{M_0}). Accordingly, upon dividing both sides by aggregate output Y (and recalling that $\psi_m = \frac{Y_M}{Y}$ is the manufacturing share of total output), we can express the conditional expected welfare loss in durables as a fraction of aggregate output as approximately equal to[6] :

[6] We note that the normalization of the relative price to unity does not affect the form of the approximate social loss function (even if the relative price were not set equal to unity, the parameter ψ_m would represent the manufacturing output share).

$$
\begin{aligned}
&\frac{1}{Y\mathbb{S}_C}\mathcal{E}_0(SW_0^M - SW_0^{M*}) = \frac{1}{Y\mathbb{S}_C}\mathcal{E}_0\sum_{t=0}^{\infty}\beta^t\left(\mathbb{W}_t^M - \mathbb{W}_t^{M\,*}\right) \approx \\
&+\psi_m\left(\tfrac{1-\delta}{2}\right)\sum_{t=0}^{\infty}\beta^t\{\mathcal{E}_0\left(y_{mt}-d_t\right)^2 - \mathcal{E}_0\left(y_{mt}^*-d_t^*\right)^2\} \\
&+\tfrac{1}{2}\psi_m\tfrac{(1-\beta(1-\delta))}{\delta\beta}\sum_{t=0}^{\infty}\beta^t\{(1-\sigma_m)(\mathcal{E}_0 d_t^2 - \mathcal{E}_0 d_t^{*2}) - \phi(\mathcal{E}_0\Delta d_{t+1}^2 - \mathcal{E}_0\Delta d_{t+1}^{*2})\} \\
&-\psi_m(1+\chi_m\ell_m)\sum_{t=0}^{\infty}\beta^t\mathcal{E}_0(g_{mt}, l_{mt}^*)\} \qquad (24) \\
&-\tfrac{1}{2}\psi_m\left(\tfrac{1+\chi_m\ell_m}{1-\alpha_m}\right)\sum_{t=0}^{\infty}\beta^t\mathcal{E}_0 g_{mt}^2 \\
&-\tfrac{1}{8}\psi_m\left(\tfrac{\theta_{Pm}}{1+\theta_{Pm}}\right)\sum_{t=0}^{\infty}\beta^t\sum_{j=0}^{3}\mathcal{E}_0\ln\left(\tilde{P}_{m,t-j}\,/\,P_{mt}\right)^2 \\
&-\tfrac{1}{8}\psi_m\left(\tfrac{1+\theta_{wm}}{\theta_{wm}}\right)(1-\alpha_m)\left(1+\left(\tfrac{1+\theta_{wm}}{\theta_{wm}}\right)\chi_m\ell_m\right)\sum_{t=0}^{\infty}\beta^t\sum_{j=0}^{3}\mathcal{E}_0\ln\left(\tilde{W}_{m,t-j}\,/\,W_{mt}\right)^2
\end{aligned}
$$

Thus, the total welfare loss attributable to both sectors (expressed as a percentage of total output) using a conditional welfare metric is simply given by the sum of equations (10) and (24). The welfare loss based on the unconditional welfare metric (i.e., $\frac{1}{Y\mathbb{S}_C}(SW - SW^*)$) used in this paper is obtained by taking the unconditional expectation of the resulting expression.[7]

[7] In computing our unconditional welfare loss measure, we abstracted from the linear "initial condition" term on the stock of durables (i.e., the term involving $d_0 - d_0^*$ in equation (23)) in order to remain within a linear-quadratic framework (the unconditional expectation of this term is zero only to a first order approximation, but not necessarily to a second order approximation). Recent computational advances (e.g., the DYNARE software) would make it possible to depart from this somewhat restrictive assumption by using a second order approximation to the model's behavioral equations, though we believe the quantitative difference would be negligible.